

Discount factor and conditional return volatility

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Using Campbell's (1991) unexpected return decomposition, the implications of the Rational Valuation Formula are derived in terms of unconditional volatility of discount factors, given conditional return volatility and hence given the volatility of unexpected returns. This provides a bound on the discount rate volatility that any econometric specification must produce in order to be admissible. Using 130 years of monthly data on the S&P Composite Index, it is shown that one needs about 10% annualized expected return volatility to explain observed conditional return volatility. The study also shows that the static and Consumption CAPM and a GARM-M specification, broadly consistent with Merton's (1973) ICAPM, produce too little discount rate variability, under a standard assumption about the degree of persistence of returns.

I. Introduction

A large body of literature has refined numerous models and techniques for estimating time-varying conditional asset price volatility, see for a review Bollersev *et al.* (1994). Another large body of literature, initiated by Shiller's (1988) seminal contribution on excess stock market volatility, has investigated at length the relation between asset price volatility and the volatility of expected fundamentals in order to check whether the former is explained by the latter as predicted by the Rational Valuation Formula (RVF) under Muth's (1961) Rational Expectation Hypothesis (RE).

Unfortunately, however, no contribution to date has focused on the implications of the RVF in terms of unconditional volatility of the discount rates, given the conditional volatility of returns and of the fundamentals. This is a pity, as it neglects to make full use, to test the RVF under RE, of the sophisticated econometric tools developed to estimate conditional time varying volatilities.

Using Campbell's (1991) unexpected return decomposition and the little variability and near unpredictability of dividend growth, I provide a simple relation that links the volatility of expected excess returns to the conditional asset price volatility and hence to the volatility of unexpected returns. I report the results of the application of this analytical framework to data on the S&P composite index for the period 1871–2003.

II. The Model

Consider Campbell's (1991) unexpected return decomposition implied by the RVF with time varying expected dividends and returns:

$$r_t - E_{t-1}(r_t) = (E_t - E_{t-1}) \left(\sum_{j=0}^{\infty} \rho^j \Delta d_{t+j} - \sum_{j=1}^{\infty} \rho^j r_{t+j} \right) \quad (1)$$

Here, r_t denotes continuously compounded rates of return, $\rho = 1/(1 + D/P)$, D and P are dividend and price at an arbitrary point (e.g., taking the typical values over a sample of US equity data for the last 50 years, $\rho = 0.96$), d_{t+j} is the log-dividend paid in $t+j$ and r_{t+j} is the rate of return between $t+j-1$ and $t+j$.

While dividends tend to grow over time, dividend growth appears to be nearly unpredictable. Dividends are therefore modelled (logged) as a random walk with a possibly non-zero deterministic drift and dividend growth (logged) as a trend-stationary process with independently and identically distributed (i.i.d.) random residuals, i.e. $d_{t+1} = d + d_t + e_{t+1}$ and $\Delta d_{t+1} = d + e_{t+1}$. Empirical evidence shows instead that returns are to some extent predictable and heteroskedastic. Therefore, I model returns as a possibly mean reverting persistent process, i.e. $r_{t+1} = \alpha + \beta r_t + \varepsilon_{t+1}$. Here, ε_{t+1} are serially uncorrelated residuals. See, for a review of the empirical evidence and a discussion, Cochrane (2001). On the basis of these assumptions about the data generating process of dividend growth and returns, Equation 1 can be rewritten as follows:

$$\begin{aligned} r_t - E_{t-1}(r_t) &= e_t - (E_t - E_{t-1}) \left(\sum_{j=1}^{\infty} \rho^j r_{t+j} \right) \\ &= e_t - \frac{\rho}{(1 - \rho\beta)} (E_t - E_{t-1}) r_{t+1} \end{aligned} \quad (2)$$

Taking the unconditional expectation of the square of both sides of Equation 2, the conditional return volatility can be approximated as follows:

$$E[r_t - E_{t-1}(r_t)]^2 \cong \frac{\rho^2}{(1 - \rho\beta)^2} E[(E_t - E_{t-1})r_{t+1}]^2 \quad (3)$$

This approximation neglects both the second moment of e_t and its cross-moment with $(E_t - E_{t-1})r_{t+1}$ because dividends are not very volatile and do not forecast returns. Solving Equation 3 for the unconditional second moment of discount rates and using the shorthand notation σ^2 for the unconditional variance of conditional residuals:

$$\begin{aligned} E[(E_t - E_{t-1})r_{t+1}]^2 &\cong E[r_t - E_{t-1}(r_t)]^2 \frac{(1 - \rho\beta)^2}{\rho^2} \\ &\cong \sigma^2 \frac{(1 - \rho\beta)^2}{\rho^2} \end{aligned} \quad (4)$$

The upshot of Equation 4 is to clarify how much variability we need in discount rates (or expected rates of return), given the unconditional volatility

of unexpected returns, returns predictability and knowledge of the typical dividend-price ratio. This result provides a useful criterion by which competing asset pricing models can be evaluated. If the model does not produce variability of discount rates of the required magnitude, it can be discarded on grounds that it will not manage to explain observed conditional price volatility. This criterion can also be used to evaluate competing reduced form econometric specifications. If a particular specification does not produce the required variability of discount rates, given estimated conditional return volatility, it cannot be considered the reduced form of an admissible (under the RVF) asset pricing model.

The return persistence parameter β will need to be estimated or calibrated to fit the data. If $\rho\beta = 1$, discount rate volatility must be zero. However, since typically $\rho < 1$, this would imply that $\beta > 1$ and, therefore, that returns are explosive. With $\beta = 0$, the required unconditional variability of discount rates is larger than the unconditional volatility of returns. In general, with $0 < \beta \leq 1$, the required variability of discount rates is lower than the volatility of returns. In particular, when returns are so persistent that they follow a random walk, i.e. $\beta = 1$, very little discount rates variability is enough to account for the volatility of returns.

III. Empirical Application

The unconditional variance σ^2 of conditional residuals in Equation 4 can be estimated by a traditional ARCH or GARCH model, see for example Engle (1982) and Bollerslev (1986), as it is simply the value to which the conditional variance of returns converges in the long run. A GARCH-M model, such as the specification proposed by Engle *et al.* (1987), would also provide estimates of the time varying conditional trade-off between expected return and risk. To be more specific, consider the following GARCH-M model of returns:

$$E_{t-1}(r_t(\theta)) = g[\sigma_{t-1}^2(\theta), \theta] \quad (5)$$

Here, a non-zero derivative of g with respect to the first element of its argument allows the conditional mean to be an explicit function of conditional variance $\sigma_{t-1}^2 = E_{t-1}(\varepsilon_t^2(\theta))$, $\varepsilon_t(\theta) = r_t - E_{t-1}(r_t(\theta))$ is a discrete time stochastic error process with conditional mean and variance parameterized by the finite dimensional vector $\theta \in \Theta \subseteq R^m$, θ_0 denotes the true value of the parameter vector. In Equation 5, the conditional mean of the residuals is $E_{t-1}(\varepsilon_t(\theta_0)) = 0$ but their conditional variance $E_{t-1}(\varepsilon_t^2(\theta_0)) = \sigma_{t-1}^2$ is

allowed to be time varying. The unconditional variance of the conditional residuals is $E(\varepsilon_t^2(\theta_0)) = \sigma^2$.

A possible empirical specification of Equation 5 is represented by a GARCH(1,1)-M model with a mean equation that includes, in addition to the conditional variance term, a constant and one lag of the stock market return:

$$\begin{aligned} E_{t-1}(r_t) &= \alpha + \alpha_1 \sigma_{t-1}^2 + \beta r_{t-1} \\ \sigma_t^2 &= \alpha_2 + \alpha_3 \varepsilon_{t-1}^2 + \alpha_4 \sigma_{t-1}^2 \end{aligned} \quad (6)$$

Conditional annualized volatilities of the S&P Composite Index¹ over the period 1871–2003 are estimated using a GARCH(1,1) specification with a first-order autoregressive model for the conditional mean. Equation 6 is therefore estimated with the restriction that $\alpha_1 = 0$. The point estimate of the β parameter is about 0.35 and the unconditional variance estimate, given by $\sigma^2 = \alpha_2 / (1 - \alpha_3 - \alpha_4)$, is about 14%. With $\beta = 0.35$ and this estimate of σ^2 , the annualized volatility of monthly discount rates implied by Equation 4 is just above 10%. Thus, one needs volatility in expected returns as large as 10% to explain the conditional return volatility over the period 1871–2003. The conditional stock market volatilities estimates from Equation 6 and the implied discount factor variability bound are plotted in Fig. 1.

The calculated 10% value for the discount rate variability bound can be used to evaluate the empirical performance of competing asset pricing models. Sharpe (1964) and Lintner's (1965) static Capital Asset Pricing Model (CAPM) is very easily discarded as it implies an obvious violation of the variability bound from below. A GARCH-M, broadly along the lines of Merton's (1973) Intertemporal CAPM (ICAPM) but with no state variable to proxy for changes in the future opportunity set, produces a yearly volatility of the discount rate of 6.15%, as shown in Fig. 1. This is computed as the sample annualized volatility of estimated conditional mean returns from Equation 6 with $\beta = 0$ but without the restriction that $\alpha_1 = 0$. The estimated variability of discount rates is still too low to satisfy the bound (it is about half the required level). Other models that price assets on the basis of their covariance with little volatile fundamental economic variables, such as aggregate consumption as in Lucas' (1978) Consumption CAPM (CCAPM), will struggle even more to reach the bound from below, unless one

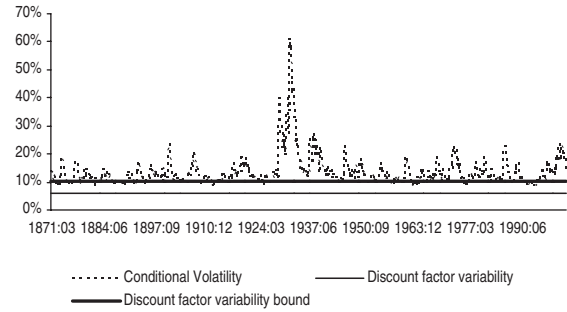


Fig. 1. Conditional return volatility vs. discount factor variability

Notes: This figure plots the conditional volatility of the S&P Composite Index (dotted line), estimated using a GARCH(1,1) with a first-order autoregressive mean equation, the implied unconditional discount factor variability bound (solid thick line) and the unconditional volatility of expected returns (solid thin line) estimated using a GARCH(1,1)-M. All volatility and variability measures are annualized. It is assumed that $\rho = 0.96$ (that corresponds to a dividend yield of 4%, the typical value found in the data). The stock market data for the period 1871–2003 used in the estimation was provided by Shiller (2001) on his website.

allows for additional sources of conditional discount factor volatility.

IV. Conclusions, Limitations and Future Research

This study derived the relation between the unconditional volatility of unexpected and expected returns, given conditional return volatility, near unpredictability of stock dividends, the typical dividend-price ratio and return persistence. This result can be used to place a bound on the volatility of conditional expected returns that, under the RVF, any econometric specification must produce to be the empirical specification of an admissible asset pricing model. The study discussed how the expected return conditional volatility bound can be used to evaluate the empirical admissibility of Sharpe (1964) and Lintner's (1965) static Capital Asset Pricing Model (CAPM), of a GARCH-M based on Merton's (1973) CAPM and of Lucas' (1978) CCAPM. A useful extension of this work would be the evaluation of a larger set of competing theoretical and econometric specifications.

¹The author also thanks Shiller (2001) for making these data available on his website.

One limitation of the derivation is that returns can only mean reversion to a constant long run level. This is implied by the proposed autoregressive specification of the return process. A useful extension would be the multivariate generalization of Equation 4 and, in particular, the derivation of the relation between conditional return volatility, conditional dividend volatility and unconditional expected return volatility when the latter are driven by a persistent, slow moving state variable. This would amount to a relatively simple generalization of the Vector Autoregressive model (VAR) presented by Cochrane (2001) to explain return predictability. To render this model able to generate conditional return volatility, it suffices to allow for heteroscedasticity in the process followed by the errors of the dividend growth and/or of the state variable. In particular, their second moments should depend, in a non-trivial fashion, on the past of the processes themselves. Campbell and Hentschel (1992) already did this for the dividend process. These developments are left for further research.

Acknowledgements

The author wishes to thank the helpful comments and suggestions provided by Colm Kearney (Trinity College Dublin), Brian Lucey (Trinity College Dublin), Liam Gallagher (Dublin City University). He also wishes to thank John Campbell (Harvard University) and John Cochrane (Chicago University) for pointing out errors in previous versions of this paper. Any remaining error is the author's own responsibility.

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