Hedge Fund Performance and Generalized Sharpe Ratios

Or Why Risk Averse Investors Love Hedge Funds

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1 Introduction

In this paper we will examine the Sharpe Ratio of hedge fund indices, and their generalized counterpart. The Generalized Sharpe Ratio has been introduced by Hodges (1998). Its advantage over the standard Sharpe Ratio is that it consistently ranks alternatives under a wide range of utility functions, rather than just the quadratic, and non-normal distributions.

This research is important because the standard Sharpe Ratio is not a reliable measure of hedge funds performance. The main reason for this is the non-normality of hedge fund returns. This in turn is due to the non-linear payoffs of the dynamic strategies pursued by such funds, that often resemble options and option-like strategies.

Section 2 contains a brief discussion of literature relevant to the paper. In section 3, we will outline the data used in our tests throughout the paper. In section 4 we will discuss the Generalized Sharpe Ratio, and apply it to the data. Finally, section 5 will conclude with a discussion of our results and limitations pertaining to them.

2 Literature Review

Sharpe (1966) introduced the Sharpe Ratio, a still widely used portfolio evaluation tool which is defined as the ratio of expected return to volatility. The use of the Sharpe Ratio has been subject to criticism. A growing literature has identified problems that limit the applicability of the Sharpe Ratio as a performance measure. The most important, outlined in Dybvig and Ingersoll (1982), is that the measure is unreliable when dealing with a portfolio which has non-linear payoffs. A related problem is that the Sharpe Ratio returns inconsistent results when the payoffs are far from normal. This has been examined by Bernardo and Ledoit (2000).

Cerny (2004) explains that there is a problem with the Sharpe ratio as a result of a fundamental flaw in the assumptions required to derive it. The Sharpe Ratio is based on the assumption that investors have a quadratic utility function:

$$U(x) = ax^2 + bx + c, \qquad a < 0 \tag{1}$$

The problem arises because for a quadratic utility function there is a "bliss" point after which any further increase in wealth would lead to a decrease in utility. In the above equation, this bliss point is:

$$x = \frac{-b}{2a}$$

This contradicts the basic assumption of expected utility theory that rational investors prefer more to less wealth. To sum up, because of the assumptions of quadratic utility, the use of the Sharpe Ratio as a performance measure leads to incorrect investment decisions depending on the level of wealth involved.

Cerny's solution to this problem is to use a so called Arbitrage-Adjusted Sharpe Ratio, which works for any level of wealth even under a quadratic utility function. Moreover, Cerny (2004), building on Hodges (1998), develops a generalized version of the standard Sharpe Ratio that is consistent with a wider class of utility functions, HARA utility. HARA (Hyperbolic Absolute Risk Aversion) utility, unlike quadratic utility, allow for preference for moments higher than the 2^{nd}. This is particularly important when we allow returns to be non multivariate normal, ie when analyzing the performance of non-linear strategies. The derivation of the Generalized Sharpe Ratio is outlined in Appendix A.

3 Data

The data used in this paper is taken from HFR's website on Hedge Fund Research (see bibliography for URL). The data set consists of 37 hedge fund indexes compiled by Hedge Fund Research. The 37 indices are listed in Appendix B. For each index we have used a 10-year time series of monthly returns beginning in January 1996 and ending in December 2005 giving 120 monthly observations for each index. Each index's return is reported net of all fees.

To compute excess returns, we used a 3-month US Treasury bill secondary market rate as a proxy for the risk free rate.[1]

Table 1 contains summary statistics for each of the 37 hedge fund indices and the market index. The S&P500 index has been used as the proxy for a market index, and its returns have been calculated monthly from January 1996 to December 2005 to coincide with the HFR data. Also shown in Table 1 are the standard Sharpe Ratios for each index, both monthly SSR and an annualized Sharpe Ratio SSR^A.[2]

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[1]The risk-free rate used is obtained by compounding the annualized rate monthly using:

$$R_f = \left((1 + R_f^a)^{1/12} - 1\right)$$

where R_f^a is the annualized 3-month Treasury bill secondary market rate.

[2]The annualized Sharpe ratio is calculated using: $SSR^A = SSR\sqrt{12}$

Table 1: **Summary statistics for excess returns of fund indices and market index.** Index number refers to numbering given in Appendix B. Monthly values for excess return mean, standard deviation, min and max figures are all given in percentages. mkt refers to the market index (S&P500) over the period. SSR denotes the calculation of the standard Sharpe Ratio. SSR^A denotes the annualized version of the sharpe ratio

Index number	obs	mean	sd	min	max	SSR	SSR^A
1	120	0.6326	2.15	-9.10	7.23	0.2945	1.0202
2	120	0.4656	0.98	-3.59	2.97	0.4728	1.6379
3	120	0.6921	1.65	-8.90	4.71	0.4193	1.4524
4	120	0.7669	4.36	-21.42	14.38	0.1760	0.6096
5	120	0.3772	3.76	-7.37	12.00	0.1004	0.3479
6	120	2.0028	8.18	-38.99	29.70	0.2449	0.8484
7	120	0.5816	4.21	-27.86	11.08	0.1381	0.4785
8	120	0.6935	5.18	-16.03	18.88	0.1338	0.4634
9	120	0.8546	2.71	-8.05	10.46	0.3154	1.0926
10	120	0.3338	0.88	-2.07	3.23	0.3812	1.3204
11	120	0.2516	1.12	-2.85	3.19	0.2238	0.7754
12	120	0.805	4.31	-13.74	10.32	0.1867	0.6468
13	120	0.7489	1.93	-9.30	4.78	0.3878	1.3435
14	120	0.3568	0.94	-3.59	2.86	0.3782	1.3103
15	120	0.1506	1.15	-6.83	2.57	0.1308	0.4531
16	120	0.4088	3.81	-11.91	14.00	0.1074	0.3720
17	120	0.3527	1.08	-2.04	4.92	0.3264	1.1308
18	120	0.3071	1.37	-7.56	2.63	0.2245	0.7776
19	120	0.4296	1.42	-9.56	3.13	0.3031	1.0499
20	120	0.3854	1.74	-7.87	6.43	0.2216	0.7678
21	120	0.3305	0.96	-4.28	3.54	0.3431	1.1885
22	120	0.3565	1.88	-8.15	7.31	0.1901	0.6584
23	120	0.4343	1.53	-5.82	4.29	0.2835	0.9822
24	120	0.4681	2.70	-12.51	9.05	0.1733	0.6002
25	120	0.5189	2.01	-4.16	6.40	0.2585	0.8954
26	120	0.7061	2.11	-3.68	5.51	0.3349	1.1601
27	120	0.4383	1.06	-6.09	2.01	0.4142	1.4348
28	120	1.0355	2.07	-4.47	8.74	0.5009	1.7353
29	120	0.4798	0.92	-6.20	2.45	0.5234	1.8131
30	120	0.7765	4.46	-13.40	17.41	0.1742	0.6035
31	120	1.5632	5.55	-12.15	19.37	0.2818	0.9763
32	120	0.8924	3.53	-19.06	10.98	0.2528	0.8757
33	120	1.0005	6.85	-18.19	41.75	0.1461	0.5062
34	120	0.4893	2.79	-8.94	7.84	0.1754	0.6075
35	120	0.5238	2.04	-7.69	6.22	0.2568	0.8894
36	120	0.9282	6.12	-15.56	21.14	0.1517	0.5255
37	120	0.019	6.32	-21.66	22.38	0.0030	0.0104
mkt	120	0.536	4.51	-14.86	9.32	0.1188	0.4116

4 The Generalized Sharpe Ratio as a Performance Measure

In order to examine how well the hedge fund indices perform, we will compare them to the market (S&P500) index and to each other on the basis of the Generalized Sharpe Ratio at various levels of risk tolerance and thus for various types of utility. Assessing performance will involve creating a simple 2-asset portfolio consisting of the index and a risk-free asset, in this case the 3-month US Treasury Bill both under a no-short sales constraint and when short sales are allowed.

Preliminarily, we we need to formally define the Generalized Sharpe Ratio and to introduce the attendant notion of investment potential. This discussion is based on the results provided by Cerny (2004). Investment potential is the percentage increase in certainty equivalent per unit of local risk tolerance. Below are five definitions of investment potential for different levels of risk tolerance.

$$IP_\gamma = \gamma\left(\left(\min_\alpha E[(1+\alpha X)^{1-\gamma}]\right)^{1/(1-\gamma)} - 1\right) for\ \gamma > 1, \quad (2)$$

$$IP_\gamma = \gamma\left(\left(\max_\alpha E[(1+\alpha X)^{1-\gamma}]\right)^{1/(1-\gamma)} - 1\right) for\ 0 < \gamma < 1, \quad (3)$$

$$IP_\gamma = exp\left(\max_\alpha E[ln(1+\alpha X)]\right) for\ \gamma = 1. \tag{4}$$

$$IP_\gamma = \gamma\left(\left(\min_\alpha E[(1+\alpha X)^{1-\gamma}]\right)^{1/(1-\gamma)} - 1\right) for\ \gamma < 0, \tag{5}$$

$$IP_{\gamma A} = \gamma\left(\left(\min_\alpha E[(\max(1+\alpha X, 0))^{1-\gamma}]\right)^{1/(1-\gamma)} - 1\right) for\ \gamma < 0, \tag{6}$$

Clearly, the solution to equations (2-6) maximizes the expected utility of investing a fraction α of wealth in the risky asset with return X. Equation (5) and equation (6) are both valid for the $\gamma < 0$ case, i.e. quadratic utility, but the latter also takes a no-arbitrage adjustment into account. The Generalized Sharpe Ratio is defined as a function of the investment potential, and thus of (2) to (6) for different types of utility within the HARA class, as explained in detail in Appendix A.

4.1 Short Sales not Allowed

Assume that the investor does not have the ability to short either the index or the treasury bill. Terminal wealth is:

$$V(\alpha_{opt}) = \alpha_{opt} V_0 R^I + (1-\alpha_{opt}) V_0 R_f, \quad 0 \le \alpha_{opt} \le 1. \tag{7}$$

In equation (7) V_0 is the initial wealth, which we can assume for simplicity to be 1, R^I is the excess return on the hedge fund index or market index, R_f is the risk-free rate and α_{opt} is the optimal portfolio weight which satisfies the minimum and/or maximum conditions in equations (2-6), depending on the assumed level of risk tolerance. The quantity α_{opt} can be seen as a measure of performance.

For now, we limit gamma to be positive and thus we consider only the case of risk averse investors (this way we also avoid the problem of getting non-real values for the Generalized Sharpe Ratios) and limit α to lie between 0 and 1 because of the short sales condition. Table 2 shows the optimal portfolio weights α_{opt} and resulting Generalized Sharpe Ratios SR for each of the indices and the market index for the full period. The level of risk tolerance assumed in each case is given in the top row.

As would be expected, in all cases the optimal value for the portfolio weight is a decreasing function of risk tolerance. That is, as we increase investors' risk tolerance, they invest less in the risky asset. More interesting is the persistence of a full weighting on the risky asset in many cases. Indeed, when $\gamma = 2$, the optimal portfolio is to invest all wealth in the risky asset for all but the worst performing index. As we increase the level of risk tolerance, less and less of the portfolios have a full weighting on the index. However, even at $\gamma = 20$, that corresponds to an investor that would be typically seen as highly risk averse, there are still 8 portfolio's which have a 100%

Table 2: Calculated optimal portfolio weights ($\alpha's$) and Generalized Sharpe Ratio's (SR) are listed for given values of risk tolerance (γ). Generalized Sharpe Ratio's are calculated using equation (25) and IP_γ is from equation (2)

γ :	2		5		10		20	
Index No.	α_{opt}	SR	α_{opt}	SR	α_{opt}	SR	α_{opt}	SR
1	1	0.1538	1	0.2297	1	0.2846	0.61	0.2907
2	1	0.1355	1	0.2121	1	0.2942	1	0.3946
3	1	0.1638	1	0.2526	1	0.3372	0.88	0.3844
4	1	0.1514	0.71	0.1704	0.36	0.1712	0.18	0.1716
5	1	0.0978	0.54	0.1014	0.27	0.1014	0.14	0.1014
6	1	0.2275	0.51	0.2366	0.26	0.2383	0.13	0.239
7	1	0.1237	0.53	0.1295	0.27	0.1304	0.14	0.1308
8	1	0.1308	0.51	0.1344	0.26	0.1344	0.13	0.1345
9	1	0.178	1	0.2638	1	0.3224	0.59	0.3269
10	1	0.1145	1	0.1787	1	0.2468	1	0.3299
11	1	0.098	1	0.1491	1	0.1958	0.97	0.2253
12	1	0.1579	0.83	0.1857	0.42	0.1861	0.21	0.1862
13	1	0.1696	1	0.259	1	0.338	0.76	0.3655
14	1	0.1183	1	0.1842	1	0.253	1	0.3307
15	1	0.0741	1	0.1074	0.92	0.1221	0.46	0.1226
16	1	0.1028	0.55	0.1073	0.28	0.1074	0.14	0.1074
17	1	0.1171	1	0.1814	1	0.247	1	0.3191
18	1	0.1076	1	0.1615	1	0.2035	0.66	0.211
19	1	0.1282	1	0.195	1	0.2506	0.68	0.2632
20	1	0.1195	1	0.1771	1	0.2165	0.59	0.2196
21	1	0.1136	1	0.1765	1	0.2412	1	0.312
22	1	0.1137	1	0.1648	0.96	0.1894	0.48	0.1896
23	1	0.1286	1	0.1955	1	0.2552	0.88	0.2848
24	1	0.126	1	0.1683	0.6	0.1712	0.3	0.1715
25	1	0.139	1	0.207	1	0.2584	0.67	0.2677
26	1	0.1636	1	0.2476	1	0.3196	0.82	0.348
27	1	0.1311	1	0.2041	1	0.2795	1	0.357
28	1	0.2009	1	0.312	1	0.4251	1	0.541
29	1	0.1378	1	0.2161	1	0.3004	1	0.4
30	1	0.1532	0.79	0.177	0.4	0.1771	0.2	0.1771
31	1	0.2273	1	0.2938	0.54	0.2945	0.27	0.2944
32	1	0.1757	1	0.236	0.58	0.2403	0.3	0.2413
33	1	0.1535	0.5	0.1559	0.25	0.1554	0.12	0.1551
34	1	0.1287	1	0.1719	0.61	0.1749	0.3	0.1751
35	1	0.1394	1	0.2066	1	0.252	0.58	0.2552
36	1	0.1509	0.51	0.1547	0.26	0.1546	0.13	0.1545
37	0.02	0.003	0.01	0.003	0	0	0	0
mkt	1	0.1151	0.51	0.1183	0.26	0.1184	0.13	0.1185

optimal weight on the risky asset. Indeed we have to increase the risk tolerance to over 40 before all portfolios have some weight on the risk-free asset.

The fund indices also seem to out-perform the market in all but a few cases. In the case of $\gamma = 2$, a total of 28 indices out-perform the market in terms of the Generalized Sharpe Ratio. The figure for the number of indices out-performing the market increases to 33 for $\gamma = 5$, and 34 for $\gamma = 10, 20$. This would suggest that, for investors with a wide range of risk profiles who do not have the ability (or permission) to short sell, holding a selection of portfolios containing hedge funds would result in a higher (Generalized) Sharpe Ratio than investing in a market index.

4.2 Short Sales Allowed

In this section we will relax the restriction on the portfolio weight in equation (7). That is, now we assume that the investor can short sell both the risk-free asset and the risky asset. However, first we must address a problem that will arise due to this restriction being relaxed. The problem is that, depending on our selection of possible values of α and γ, we may get non-real values for the Sharpe Ratio. To obtain the explicit formula for calculating the Sharpe Ratio, we must substitute equation (2) into equation (25). In the case of $\gamma > 1$, we get:

$$SR_\gamma = \left(\left(\min_\alpha E[(1+\alpha X)^{1-\gamma}] \right)^{\frac{2\gamma}{1-\gamma}} - 1 \right)^{\frac{1}{2}} \tag{8}$$

Due to the square root, in order to assure that we get a real value for the Sharpe Ratio, we must impose the following restriction:

$$\begin{aligned} & (\min_\alpha E[(1+\alpha X)^{1-\gamma}])^{\frac{2\gamma}{1-\gamma}} - 1 & > 0 \\ \Rightarrow \quad & (\min_\alpha E[(1+\alpha X)^{1-\gamma}])^{\frac{2\gamma}{1-\gamma}} & > 1 \end{aligned} \tag{9}$$

Without the restriction in equation (9), we can see that for any choice of γ, there will be values of α that will return Generalized Sharpe Ratios with imaginary parts. We now define the Generalized Sharpe Ratio for all values of $\gamma \in \Re$, where $\Re$ is the set of real numbers, using equations (2-6) and the restrictions specified above.

$$SR_\gamma = \left(\left(\min_\alpha E[(1+\alpha X)^{1-\gamma}] \right)^{\frac{2\gamma}{1-\gamma}} - 1 \right)^{\frac{1}{2}} \; for \; \gamma > 1, \tag{10}$$

$$\forall \; \alpha \; st \quad (\min_\alpha E[(1+\alpha X)^{1-\gamma}])^{\frac{2\gamma}{1-\gamma}} > 1$$

$$SR_\gamma = \left(\left(\max_\alpha E[(1+\alpha X)^{1-\gamma}] \right)^{\frac{2\gamma}{1-\gamma}} - 1 \right)^{\frac{1}{2}} \; for \; 0 < \gamma < 1, \tag{11}$$

$$\forall\, \alpha\ st \quad (\max_\alpha E[(1+\alpha X)^{1-\gamma}])^{\frac{2\gamma}{1-\gamma}} > 1$$

$$SR_\gamma = \left(\left(1 + \exp\left(\max_\alpha E[ln(1+\alpha X)]\right)\right)^2 - 1\right)^{\frac{1}{2}} \; for\ \gamma = 1, \quad (12)$$

$$\forall\, \alpha\ st \quad 1+\alpha X > 0$$

$$SR_\gamma = \left(\left(\min_\alpha E[(1+\alpha X)^{1-\gamma}]\right)^{\frac{2\gamma}{1-\gamma}} - 1\right)^{\frac{1}{2}} \; for\ \gamma < 0, \qquad (13)$$

$$\forall\, \alpha\ st \quad (\min_\alpha E[(1+\alpha X)^{1-\gamma}])^{\frac{2\gamma}{1-\gamma}} > 1$$

$$SR_{\gamma A} = \left(\left(\min_\alpha E[(\max(1+\alpha X, 0))^{1-\gamma}]\right)^{\frac{2\gamma}{1-\gamma}} - 1\right)^{\frac{1}{2}} \; for\ \gamma < 0, \qquad (14)$$

$$\forall\, \alpha\ st \quad (\min_\alpha E[(\max(1+\alpha X, 0))^{1-\gamma}])^{\frac{2\gamma}{1-\gamma}} > 1$$

Now that we have assured that the Generalized Sharpe Ratio takes only real values, we can apply equations (10-14) to our hedge fund and market index data allowing for negative risk tolerance and negative weights on either of the assets. While it is conceivable that an investor could possibly short an unlimited (or almost unlimited) amount of an asset, we must concede that there is a limit (whether imposed on the investor externally or otherwise) to the amount of an asset an investor will reasonably be shorting. For the purpose of our tests, we will assume that an investor will only be able to (or want to) short 5 times his wealth in any asset, whether it be the risk-free asset, a hedge fund index, or a market index. This means that for our test we have the added restriction of $-5 \leq \alpha \leq 6$.

Tables 3, 4 and 5 contain the calculations for the optimal weights and resulting generalized Sharpe Ratios for a wide range of risk tolerance levels. They roughly correspond to risk tolerance levels of risk averse, risk neutral and risk loving investors. The case $\gamma = -5$ implies highly risk loving preferences whereas $\gamma = 9$ implies relatively high risk aversion.

Table 3 gives the Generalized Sharpe Ratios for risk averse investors. The results are similar to the results in table (2) in that the majority of the fund indices take a large weighting in the portfolios (600% in some cases). Relaxing the restriction on alpha and in particular on the extent to which we can short the risk-free asset, the optimal weight of the fund index within some of the portfolios reaches as much as 2900%. We again see the majority of fund indices outperforming the market index in terms of their Generalized Sharpe Ratios. In fact, we find that more of the fund indices have higher Generalized Sharpe Ratios than the market index. In the case of $\gamma = 2$, 35 of the 37 hedge fund indices have higher Generalized Sharpe Ratios than the market. At $\gamma = 5$, 34 of the hedge fund indices outperform the market and this is the same for the $\gamma = 9$ case. In the three cases above, the same 34 fund indices outperformed the market each time, with one extra index beating the

Table 3: Calculated optimal portfolio weights ($\alpha's$) and Generalized Sharpe Ratio's (SR) are listed for given values of risk tolerance (γ). Generalized Sharpe Ratio's are calculated using equation (10). Dashes (-) denote non-real Sharpe Ratios

γ :	1.5		2		5		9	
Index No.	α_{opt}	SR	α_{opt}	SR	α_{opt}	SR	α_{opt}	SR
1	6	0.2728	5.43	0.2807	2.37	0.2879	1.35	0.2896
2	6	0.2788	6	0.3176	6	0.4434	4.17	0.4575
3	6	0.3182	6	0.3474	3.24	0.3741	1.89	0.3803
4	-	-	4.69	0.999	0.71	0.1704	0.4	0.1711
5	1.81	0.1015	1.36	0.1015	0.54	0.1014	0.3	0.1014
6	-	-	5.74	0.888	0.51	0.2366	0.29	0.2381
7	-	-	3.61	4.2525	0.53	0.1295	0.3	0.1303
8	1.68	0.1337	1.27	0.134	0.51	0.1344	0.29	0.1344
9	6	0.314	5.63	0.3228	2.35	0.3261	1.31	0.3266
10	6	0.2346	6	0.2669	6	0.3761	4.91	0.3995
11	6	0.1884	6	0.2069	3.88	0.2249	2.16	0.2252
12	2.66	0.1838	2.03	0.1846	0.83	0.1857	0.46	0.186
13	6	0.3197	6	0.3411	2.86	0.3582	1.65	0.3626
14	6	0.2403	6	0.272	6	0.3542	3.43	0.3576
15	5.35	0.117	4.18	0.1185	1.79	0.1212	1.01	0.122
16	1.81	0.1067	1.37	0.1069	0.55	0.1073	0.31	0.1074
17	6	0.2357	6	0.2658	6	0.349	3.78	0.3512
18	6	0.1943	5.74	0.202	2.52	0.2082	1.44	0.2099
19	6	0.2339	5.4	0.243	2.52	0.2565	1.46	0.2604
20	6	0.2091	5.39	0.214	2.31	0.2181	1.3	0.219
21	6	0.2296	6	0.259	5.72	0.3294	3.27	0.3324
22	5.86	0.1847	4.54	0.1863	1.91	0.1887	1.07	0.1893
23	6	0.2451	6	0.2672	3.43	0.2831	1.93	0.2842
24	3.65	0.1669	2.83	0.1684	1.18	0.1706	0.67	0.1712
25	6	0.2519	6	0.2661	2.7	0.2679	1.5	0.2678
26	6	0.3086	6	0.3347	3.28	0.3484	1.82	0.3482
27	6	0.2648	6	0.298	4.64	0.3571	2.7	0.3633
28	6	0.404	6	0.4555	5.5	0.5656	3.07	0.5658
29	6	0.2836	6	0.3225	5.22	0.4138	3.1	0.4247
30	2.59	0.1761	1.96	0.1765	0.79	0.177	0.44	0.1771
31	3.51	0.2933	2.67	0.2942	1.08	0.2947	0.6	0.2945
32	-	-	5.27	0.6357	1.13	0.2379	0.64	0.24
33	1.7	0.1575	5.52	0.5009	0.5	0.1559	0.28	0.1555
34	3.87	0.1727	2.95	0.1735	1.21	0.1746	0.67	0.1749
35	6	0.2439	5.4	0.2495	2.29	0.2536	1.29	0.2546
36	1.7	0.1546	-4.89	1.0505	0.51	0.1547	0.28	0.1546
37	0.03	0.003	-4.49	2.5064	0.01	0.003	0.01	0.0015
mkt	1.67	0.1175	1.26	0.1178	0.51	0.1183	0.29	0.1184

Table 4: Calculated optimal portfolio weights ($\alpha's$) and Generalized Sharpe Ratio's (SR) are listed for given values of risk tolerance (γ). Generalized Sharpe Ratio's are calculated using equations (11) and (12). Dashes (-) denote non-real Sharpe Ratios

γ :	1		.5	
Index No.	α_{opt}	SR	α_{opt}	SR
1	6	0.2432	6	0.1836
2	6	0.2303	6	0.1645
3	6	0.2706	6	0.1972
4	-	-	-	-
5	2.71	0.1014	5.31	0.1009
6	-	-	-	-
7	-	-	-	-
8	2.48	0.1331	4.53	0.1301
9	6	0.2801	6	0.212
10	6	0.1942	6	0.1391
11	6	0.1607	6	0.1182
12	3.85	0.1819	6	0.1729
13	6	0.276	6	0.2034
14	6	0.1998	6	0.1435
15	6	0.1111	6	0.0876
16	2.67	0.1062	4.95	0.104
17	6	0.1967	6	0.142
18	6	0.1716	6	0.1288
19	6	0.2061	6	0.1536
20	6	0.1876	6	0.1427
21	6	0.1913	6	0.1378
22	6	0.1727	6	0.1348
23	6	0.2101	6	0.1548
24	5.06	0.1634	6	0.1444
25	6	0.2213	6	0.1663
26	6	0.2656	6	0.1963
27	6	0.2208	6	0.1588
28	6	0.3366	6	0.2421
29	6	0.2343	6	0.1672
30	3.76	0.1748	6	0.1677
31	5.04	0.2905	6	0.2598
32	-	-	-	-
33	2.55	0.158	4.59	0.1562
34	5.6	0.1709	6	0.1487
35	6	0.2187	6	0.1662
36	2.52	0.1542	4.64	0.1515
37	0.05	0.003	0.1	0.003
mkt	2.46	0.1169	4.54	0.1143

Table 5: Calculated optimal portfolio weights ($\alpha' s$) and Generalized Sharpe Ratio's (SR) are listed for given values of risk tolerance (γ). Generalized Sharpe Ratio's are calculated using equation (14).

γ :	-.5		-1		-2		-5	
Index No.	α_{opt}	SR	α_{opt}	SR	α_{opt}	SR	α_{opt}	SR
1	-5	0.1719	-5	0.2314	-5	0.2879	-2.55	0.2942
2	-5	0.1523	-5	0.2135	-5	0.2961	-5	0.429
3	-5	0.1844	-5	0.2554	-5	0.3424	-3.91	0.3994
4	-5	0.1647	-3.95	0.1767	-1.93	0.175	-0.75	0.1734
5	-5	0.0997	-2.67	0.1008	-1.35	0.1012	-0.54	0.1013
6	-5	0.2373	-2.85	0.2459	-1.42	0.2449	-0.55	0.2423
7	-5	0.1371	-3.25	0.1387	-1.52	0.1354	-0.58	0.1331
8	-4.76	0.1322	-2.56	0.1343	-1.29	0.1346	-0.52	0.1346
9	-5	0.1985	-5	0.2646	-5	0.3215	-2.38	0.3269
10	-5	0.1284	-5	0.1795	-5	0.2476	-5	0.3572
11	-5	0.1092	-5	0.1495	-5	0.1962	-3.92	0.2257
12	-5	0.1709	-4.22	0.1875	-2.12	0.1873	-0.84	0.1868
13	-5	0.1907	-5	0.262	-5	0.3436	-3.29	0.3756
14	-5	0.1326	-5	0.1852	-5	0.2544	-5	0.3542
15	-5	0.0823	-5	0.1089	-5	0.1273	-1.95	0.1248
16	-5	0.1067	-2.81	0.1078	-1.41	0.1078	-0.56	0.1076
17	-5	0.1311	-5	0.182	-5	0.2474	-5	0.3369
18	-5	0.12	-5	0.1628	-5	0.2071	-2.79	0.2153
19	-5	0.1438	-5	0.1974	-5	0.2578	-3.03	0.2741
20	-5	0.1329	-5	0.1779	-5	0.2185	-2.44	0.2216
21	-5	0.1273	-5	0.1773	-5	0.2423	-5	0.3318
22	-5	0.1259	-5	0.1653	-5	0.1914	-1.98	0.1907
23	-5	0.1437	-5	0.1964	-5	0.2562	-3.58	0.2866
24	-5	0.1384	-5	0.1702	-3.14	0.1736	-1.23	0.1727
25	-5	0.1547	-5	0.2074	-5	0.2574	-2.67	0.2668
26	-5	0.1831	-5	0.2489	-5	0.3196	-3.22	0.3466
27	-5	0.1472	-5	0.2056	-5	0.2821	-5	0.3802
28	-5	0.2268	-5	0.315	-5	0.4269	-5	0.5587
29	-5	0.155	-5	0.2179	-5	0.3032	-5	0.4375
30	-5	0.1621	-3.83	0.175	-1.98	0.1766	-0.8	0.1771
31	-5	0.2446	-4.74	0.283	-2.6	0.2905	-1.07	0.2932
32	-5	0.1953	-5	0.2444	-3.3	0.2505	-1.26	0.2462
33	-3.54	0.1346	-2.11	0.1468	-1.17	0.1515	-0.49	0.1537
34	-5	0.1412	-5	0.1729	-3.09	0.1761	-1.23	0.1757
35	-5	0.1553	-5	0.2078	-5	0.2542	-2.39	0.2571
36	-4.3	0.1474	-2.44	0.1523	-1.26	0.1537	-0.51	0.1543
37	-0.1	0.003	-0.05	0.003	-0.02	0.003	-0.01	0.003
mkt	-5	0.1192	-2.62	0.1193	-1.3	0.119	-0.52	0.1188

market at the lower level of risk tolerance.

For risk neutral investors (Table 4), results are similar. The optimal portfolio is to be as long as possible (ie:600%) on hedge fund indices and as short as possible (ie:500%) on the risk-free asset. This is the case for 29 of the fund indices when $\gamma = 1$ and 27 of the fund indices when $\gamma = .5$. Only 4 of the 33 fund indices with real Generalized Sharpe Ratios do not beat the market in therms of this ratio when $\gamma = 1$ and only 3 failed to beat the market portfolio when $\gamma = .5$.

Table 5 shows data for risk loving investors. For each risk tolerance level, the number of portfolios containing hedge fund indices with higher Generalized Sharpe Ratios than the market is similar to the cases featuring risk averse and risk neutral investors. However, and not surprisingly, the optimal weights on the risky assets, including the market index, are negative for every portfolio.

5 Conclusion

Our results indicate that, based on an arbitrage adjusted and Generalized Sharpe Ratio, hedge funds outperform the market for non risk loving investors with any type of preferences within the HARA utility class. However, there are some limitations in our analysis. Firstly, we used hedge fund index rather than individual hedge fund data. An implication of this is that there might be an averaging out of charachteristics like the volitility and other moments of the distribution of returns. Volatility of our indices almost certainly underestimates the volatility of each individual hedge fund. For example, we saw that as we increased γ and thus as we considered more risk averse investors, α slowly declined towards 0. This gradual decay may be more pronounced in the case of individual hedge funds. Also, the tests only take into account a 10-year sample period. It is difficult to draw very strong conclusions from data spanning such a short time frame. It could be argued that the hedge fund out-performance might be the result of a bubble or of a 'peso' problem in the estimation of mean returns. As data on longer sample periods become available, further research might therefore re-examine this issue and perhaps attempt to use more efficient techniques to estimate the mean return offered by strategies involving hedge funds.

Appendix A

Derivation of the Generalized Sharpe Ratio

The derivation follows Cerny (2004). We begin with the equation for the end-of-period wealth for a person's investment(s):

$$V(\tilde{\alpha}) = \tilde{\alpha} V_0 \hat{R}' + (1 - \tilde{\alpha}) V_0 R_f + y \tag{15}$$

where:

V is the end-of-period wealth

$\tilde{\alpha}$ is the proportion of wealth invested in risky assets

V_0 is the initial wealth

$\hat{R}$ is the return on the risky assets

R_f is the return from a safe asset

y is the income

In this paper, I will be assuming the HARA (Hyperbolic Absolute Risk Aversion) class of utility functions are appropriate. We use this since we want generality, and the HARA class can be specified to correspond to CARA, CRRA or quadratic utility functions as needs be. We define the general HARA Utility with baseline risk aversion γ and local risk aversion $\hat{\gamma}$ as:

$$U_{\gamma,\tilde{\gamma}}(V(\alpha_{\gamma,\tilde{\gamma}})) = \frac{V_{safe}^{1-\gamma}}{1-\gamma} \left(\frac{\gamma}{\tilde{\gamma}} + \alpha_{\gamma,\tilde{\gamma}} X \right)^{1-\gamma} \tag{16}$$

where

$\alpha_{\gamma,\tilde{\gamma}}$ is the proportion of wealth invested in risky assets
V_{safe} is the end-of-period risk-free wealth
X is the excess return (ie: $\hat{R} - R_f$)

We now define the Investment Potential (IP) to be the increase in certainty equivalent ($V_{certain}$) per unit of local risk tolerance. We have:

$$IP_{\gamma}(\alpha_{\gamma,1}, X) = \left(\frac{V_{\gamma,1\,certain}(\alpha_{\gamma,1})}{V_{safe}} - 1 \right) \quad (17)$$

The maximum investment potential can then be defined to be:

$$IP_{\gamma}(X) = \max_{\alpha} IP_{\gamma}(\alpha, X) \quad (18)$$

We seek to generalize the Sharpe Ratio and write it in the form of the investment potential. We must begin by first deriving this for the quadratic utility. The quadratic utility is a specific case of the HARA utilities with $\gamma = -1$ and $\hat{\gamma} = V_{safe}/(V_{bliss} - V_{safe})$. Rewriting (15), we get:

$$V(\alpha) = (V_0 R_f + y)(1 + \alpha X) \quad (19)$$

Also, equation (1) can be written in a different form:

$$U(V) = -\frac{1}{2}(V - V_{bliss})^2$$

Using this and (19), we wish to maximize the expected quadratic utility:

$$-E[(V_{bliss} - V_{safe}(1 + \alpha X))^2] \qquad (20)$$

We can go on to show that: (for more detailed derivation see Cerny (2004) pg:67)

$$\left(\frac{V_{-1,\hat{\gamma}\,certain}(\alpha_{-1,\hat{\gamma}})}{V_{safe}} - 1\right) = \frac{1}{\hat{\gamma}}(1 - (E[(1 - \hat{\gamma}\alpha_{-1,\hat{\gamma}}X)^2])^{\frac{1}{2}}) \qquad (21)$$

the investment potential.

In order to find the optimal investment assuming quadratic utility, we must maximize the Investment Potential in equation (21). This problem can be written as:

$$\begin{aligned} &\max_{\alpha}(1 - (E[1 - \alpha X]^2)^{1/2}) \\ &\equiv \min_{\alpha}(E[1 - \alpha X]^2)^{1/2} \end{aligned}$$

This is equivalent to maximizing the normalized quadratic utility:

$$\min_{\alpha} u(\alpha) = \min_{\alpha} E[(1-\alpha X)^2] \tag{22}$$
$$\Rightarrow u'(\alpha) = \frac{\partial}{\partial \alpha} E[1 - 2\alpha X + \alpha^2 X^2] = 0$$
$$\Rightarrow \frac{\partial}{\partial \alpha}(1 - 2\alpha E[X] + \alpha^2 E[X^2]) = 0$$
$$\Rightarrow -2E[X] + 2\alpha E[X^2]) = 0 \Rightarrow \alpha_{-1,1\,opt} = \frac{E[X]}{E[X^2]}$$

Substituting α_{opt} into equation (21), we get the maximum investment potential:

$$IP_{-1}(X) = 1 - \left(1 - \frac{(E[X])^2}{E[X^2]}\right)^{1/2} \tag{23}$$

So assets with the same value of $1 - \frac{(E[X])^2}{E[X^2]}$ as above will have the same increase (or decrease) in quadratic utility. Rearranging this we get:

$$\begin{aligned} 1 - \frac{(E[X])^2}{E[X^2]} &= \frac{E[X^2]-(E[X])^2}{E[X^2]-(E[X])^2+(E[X])^2} \\ &= \frac{1}{1+\frac{(E[X])^2}{E[X^2]-(E[X])^2}} \\ &= \frac{1}{1+SR^2(X)} \end{aligned} \tag{24}$$

The previous step comes from the standard definition of the Sahrpe Ratio:

$$SR = \frac{E[X]}{\sqrt{E[X^2] - (E[X])^2}}$$

Substituting equation (24) into equation (22), we get:

$$\min_{\alpha_{-1,1}} E[(1 - \alpha_{-1,1}X)^2] = \frac{1}{1 + SR^2(X)}$$

Finding and inputting the optimal value for $\alpha_{-1,1}$ will give us a value for the maximum Sharpe Ratio. However, this just for quadratic utility functions. Rewriting the above and generalizing we get:

$$1 + SR_\gamma^2 = \left(1 + \frac{IP_\gamma}{\gamma}\right)^{2\gamma}$$
$$\Rightarrow SR_\gamma = \left[\left(1 + \frac{IP_\gamma}{\gamma}\right)^{2\gamma} - 1\right]^{1/2} \tag{25}$$

Equation (25) is our definition of the Generalized Sharpe Ratio for risk tollerance γ.

Appendix B

Fund Indices used in Data

Table 6: List of 37 funds indices used as data

Fund Index No.	Name of Fund Index
1	HFRI Fund Weighted Composite Index
2	HFRI Convertible Arbitrage Index
3	HFRI Distressed Securities Index
4	HFRI Emerging Markets (Total)
5	HFRI Emerging Markets: Asia Index
6	HFRI Emerging Markets: Eastern Europe/CIS Index
7	HFRI Emerging Markets: Global Index
8	HFRI Emerging Markets: Latin America Index
9	HFRI Equity Hedge Index
10	HFRI Equity Market Neutral Index
11	HFRI Equity Market Neutral Index: Statistical Arbitrage
12	HFRI Equity Non-Hedge Index
13	HFRI Event-Driven Index
14	HFRI Fixed Income (Total)
15	HFRI Fixed Income: Arbitrage Index
16	HFRI Fixed Income: Convertible Bonds Index
17	HFRI Fixed Income: Diversified Index
18	HFRI Fixed Income: High Yield Index
19	HFRI Fixed Income: Mortgage-Backed Index
20	HFRI Fund of Funds Composite Index
21	HFRI FOF: Conservative Index
22	HFRI FOF: Diversified Index
23	HFRI FOF: Market Defensive Index
24	HFRI FOF: Strategic Index
25	HFRI Macro Index
26	HFRI Market Timing Index
27	HFRI Merger Arbitrage Index
28	HFRI Regulation D Index
29	HFRI Relative Value Arbitrage Index
30	HFRI Sector (Total)
31	HFRI Sector: Energy Index
32	HFRI Sector: Financial Index
33	HFRI Sector: Health Care/Biotechnology Index
34	HFRI Sector: Miscellaneous Index
35	HFRI Sector: Real Estate Index
36	HFRI Sector: Technology Index
37	HFRI Short Selling Index

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