Models of Asset Dynamics

Valerio Potì

November 2008

- Two main models additive model $S(k+1) = aS(k) + u(k)$

  multiplicative model $S(k+1) = S(k)u(k)$

  here,

  $S(k)$ asset price in k
  $u(k)$ zero mean **random** asset price innovation in k, $E(u(k))=0$
  $k = 0, 1, 2, \ldots, N-1$

- Additive model

  S(k+1) = aS(k) + u(k)

  u(k) = S(k+1) - S(k) — price change or capital gain between k and k+1

  S(1) = aS(0) + u(0)
  S(2) = aS(1) + u(1)
  $= a^2S(0) + au(0) + u(1)$
  ......
  $S(k) = a^kS(0) + a^{k-1}u(0) + a^{k-2}u(1) + ... + u(k-1)$

  hence S(k) is the sum of a deterministic part, $a^kS(0)$, and of k random disturbances

  Expected value of S grows geometrically at a **drift rate** *a*

  $$E[S(k)] = a^kS(0)$$

  The additive model has the drawback that price can become **negative**!

  - If u(k) are iid, (almost) any average of a *large sample* of them is normally distributed
    - ✓ iid means that u(0), u(1),....u(N-1) are mutually independent random variables

  - Thus, a linear combination and especially their sum should be normally distributed too.

  If u(k) are iid with variance σ^2, for small time increments:

  ⇨ $a^{k-1}u(0) + a^{k-2}u(1) + ... + u(k-1) \sim N(0, \sigma^2/(1-a))$
  ⇨ $S(k) \sim N(a^kS(0), \sigma^2/(1-a))$

  or, when a = 1

  ⇨ $u(0) + u(1) + ... + u(k-1) \sim N(0, k\sigma^2)$
  ⇨ $S(k) \sim N(a^kS(0), k\sigma^2)$

  - You get the same results if you directly assume $u(k) \sim N(0, \sigma^2)$

- The multiplicative model:

$S(k+1) = S(k)u(k)$

$u(k) = S(k+1)/S(k)$ — price growth or total return (neglecting dividends) between k and k+1

$S(1) = S(0)u(0)$
$S(2) = S(1)u(1)$
$\quad = S(0)\, u(0)\, u(1)$
......
$S(k) = S(0)\, u(0)\, u(1) \,.....\, u(k-1)$

Take logs:

$\ln[S(k+1)] = \ln[S(k)u(k)] = \ln S(k) + \ln u(k)$ — an additive model!

Define <u>*log-growth rates*</u> $\boldsymbol{w}\mathbf{(k) = \ln u(k)}$

Notice
✓ $w(k-1) = \ln S(k)/S(k-1)$
$\quad = \ln S(k) - \ln S(k-1)$ ($= \Delta \ln S$)

$\quad = \ln [S(k-1)e^r / S(k)]$
$\quad = \ln [S(k-1)e^r] - \ln S(k-1)$
$\quad = \ln e^r$

$\quad = r$ — the continuously compounded rate of return! (with no dividends)

Again, if <u>**either**</u> the time increments are small and the $w(k) = \ln u(k)$ are iid with mean $\overline{w}(k) = v$ and variance σ^2 <u>**or**</u> $w(k) \sim N(v, \sigma^2)$, the price is **log-normally** distributed:

⇨ $\ln u(0) + \ln u(1) + ... + \ln u(k) \sim N(vk, k\sigma^2)$
⇨ $\ln S(k) \sim N(\ln S(0) + vk, k\sigma^2)$

Notice:

$$u(k) = e^{w(k)} = e^{\ln u(k)} > 0 \quad \forall \quad w(k)$$

Thus, when $S(0) \geq 0$,

$$S(k) = S(0)\prod_{i=0}^{k-1} u(i) \geq 0 \quad \forall \quad k$$

i.e., PRICE IS NEVER NEGATIVE

- Log-normal random variables

$$u(k) = e^{w(k)}$$

$$\bar{u}(k) = e^{\bar{w}(k)} \,?$$

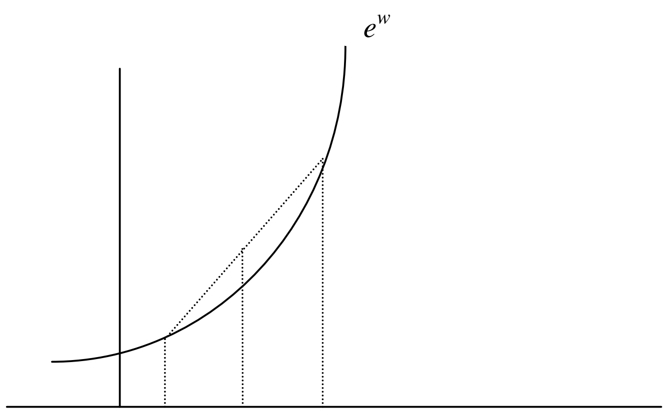
Almost right!

.....since the exponential function is convex its average is larger than the average of the argument (see Figure below)

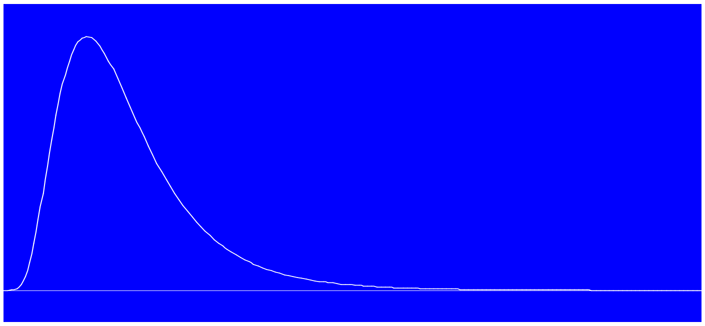
$$\bar{u}(k) = e^{\bar{w}(k)+\frac{1}{2}\sigma^2} = e^{v(k)+\frac{1}{2}\sigma^2} = e^{\mu}$$

Where,

$$\mu = v(k) + \frac{1}{2}\sigma^2$$



For formal proof we need Ito's Lemma…..see later



$$E[S(t)] = S_0 E\left(\prod_{i=0}^{t-1} u_i\right)$$

Then, because the u_i are iid,

$$= S_0 \prod_{i=0}^{t-1} E(u_i)$$

$$= S_0 \prod_{i=0}^{t-1} \overline{u}(t)$$

$$= S_0 e^{\mu t}$$

Also,

$$\operatorname{var}[S(t)] = S_0{}^2 e^{2\mu t} (e^{\sigma^2 t} - 1)$$

Typical parameter values:

$v = ?$
$\sigma^2 = ?$

From historical data (with caution!), with N periods and N+1 price observations:

$$
\begin{aligned}
\hat{v} &= \overline{w} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} w(k) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \left[\ln S(k+1) - \ln S(k)\right] \\
&= \frac{1}{N}\left[\ln S(N) - \ln S(0)\right] \\
&= \frac{1}{N}\left[\ln \frac{S(N)}{S(0)}\right] \\
&= \frac{1}{N}\ln R
\end{aligned}
$$

and,

$$\text{var}(\hat{v}) = \frac{1}{N}\sigma^2$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{k=0}^{N-1}\left[w(k) - \hat{v}\right]^2$$

$$\text{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{N-1}$$

- For periods k of 1 year each, v = 12% and σ = 16% (so, $\sigma^2 \cong 2.5\%$) using NYSE long-run data

- Hence, we need roughly 10 years of annual data to reduce the standard deviation of the estimates to 5% (still a sizeable fraction of the historical average)

$$\mathrm{var}(\hat{v}) = \frac{1}{N}\sigma^2 = \frac{1}{10}16\%^2 \cong \frac{1}{10}2.5\% = 0.25\%$$
$$stdev(\hat{v}) = \sqrt{\mathrm{var}(\hat{v})} = \sqrt{0.25\%} \cong 5\%$$

- But with the estimates of variance things get better:

$$\mathrm{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{N-1} = \frac{2(16\%)^4}{9} = \frac{0.13\%}{9} = 0.015\%$$
$$stdev(\hat{\sigma}^2) = \sqrt{0.015\%} = 1.21\%$$

- Annual growth rates and variance scale proportionally and volatility scale according to the square root of time.

- If there are p periods in a year:

$$v_p = pv \qquad \sigma_p^2 = p\sigma^2 \qquad \sigma_p = \sqrt{p}\sigma$$

Example: if the data is monthly (p = 12) and there are T = 10 years of data, $N = pT = 120$, v is the average monthly growth rate , v_{12} is its annualized value and σ_{12} is its annualized volatility.

Thus, given the historical estimates of $\hat{v}_{12}$ and $\hat{\sigma}_{12}$ from annual data, the monthly average growth rate and volatility estimates are

$$\hat{v} = \frac{12\%}{p} = 1\% \qquad \text{and} \qquad \hat{\sigma} \cong \frac{16\%}{3.46} = 4.6\%$$

Random Walk and Wiener Processes

- both functions of time

- Random walks

  Suppose we have N periods of length Δt

  $$z(t_{k+1}) = z(t_k) + \varepsilon(t_k)\sqrt{\Delta t}$$
  $$t_{k+1} = t_k + \Delta t$$

  z is a random walk

  - this is an additive process
  - $\varepsilon(t_k) \sim N(0,\ 1)$
  - $E[\varepsilon(t_k)\varepsilon(t_j)] = 0 \quad \forall\, k \neq j$
  - i.e. $\varepsilon(t_k)$ is a iid standardized normal random variable
  - can simulate a random walk by setting initial value (t_0) and then adding random iid increments

Cumulated increments of random walks:

$$z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}$$

$$E\left[z(t_k) - z(t_j)\right] = 0$$

$$\begin{aligned}
\text{var}\left[z(t_k) - z(t_j)\right] &= E\left[\sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}\right]^2 \\
&= E\left[\sum_{i=j}^{k-1} \varepsilon(t_i)^2 \Delta t\right] \\
&= \sum_{i=j}^{k-1} E\left[\varepsilon(t_i)^2\right]\Delta t \\
&= \sum_{i=j}^{k-1} 1\Delta t \\
&= \sum_{i=j}^{k-1} \Delta t \\
&= (k - j)\Delta t \\
&= t_k - t_j
\end{aligned}$$

The variance of the cumulated increments is exactly equal to the time elapsed!

Also,

$$E\left\{\left[z(t_k) - z(t_j)\right]\left[z(t_m) - z(t_l)\right]\right\} = 0 \quad i = k....j \neq s = m...l$$

- Weiner Processes

$$\lim_{\Delta t \to 0} \Delta z = dz$$

If the limit is defined, z is a Weiner process (or Brownian motion) and

  - $dz = \lim_{\Delta t \to 0} \Delta z = \lim_{\Delta t \to 0} \left[z(t_k) - z(t_j) \right] = \varepsilon(t)\sqrt{dt}$
  - $\varepsilon(t) \sim N(0,\ 1)$
  - $E\left[\varepsilon(t')\varepsilon(t'')\right] = 0 \quad \forall t' \neq t''$
  - i.e. $\varepsilon(t)$ is a iid standardized normal random variable

- So, the mean and variance of dz are

$$E(dz) = 0$$
$$\text{var}(dz) = \text{var}\left[\varepsilon(t)\sqrt{dt}\right] = dt\ \text{var}[\varepsilon(t)] = dt$$

- Notice: the variance of dz/dt is infinite

$$E\left[\frac{z(s) - z(t)}{s - t}\right]^2 = \frac{s - t}{(s - t)^2} = \frac{1}{s - t} \xrightarrow{s \to t} \infty$$

- Notice: since the variance of dz/dt is infinite, dz is not differentiable w.r.t. time
- Yet, the term dz/dt is very used in stochastic calculus; it is called **white noise**

- Generalized Weiner Processes

  $dx(t) = a\,dt + b\,dz$
  - $x(t)$ is a random variable
  - z is a Wiener process
  - a and b are constants

  To find the 'solution', integrate both sides:

$$\int dx = x + c$$

$$\int (adt + bdz) = \int adt + \int bdz = a\int dt + b\int dz = a(t+c) + b(z+c)$$

  Thus,

$$x + c = at + ac + bz + bc$$

  What about the arbitrary constant c? To get rid of it, let

$$x = x(0) + at + bz$$

  Simplifying and solving for c, this implies

$$c - ac - bc = x(0)$$

$$\Rightarrow c = \frac{x(0)}{1-a-b}$$

  Thus, $x = x(0) + at + bz$ is a legitimate 'solution' when $a + b \neq 1$. More accurately, the 'solution' is:

$$x(t) = x(0) + a\,t + b\,z(t)$$

  To double check take first derivative of $x(t)$

$$dx(t)/dt = a + b\,dz(t)/dt$$

  or

$$dx(t) = a\,dt + b\,dz(t) \qquad \text{checked!}$$

- Ito Processes

$dx(t) = a(x, t)\, dt + b(x, t)\, dz$

- $x(t)$ is a random variable
- z is a Wiener process
- a and b are parameter that may depend on x and t

This process does not have a general analytic 'solution'.

Price Dynamics Modeling

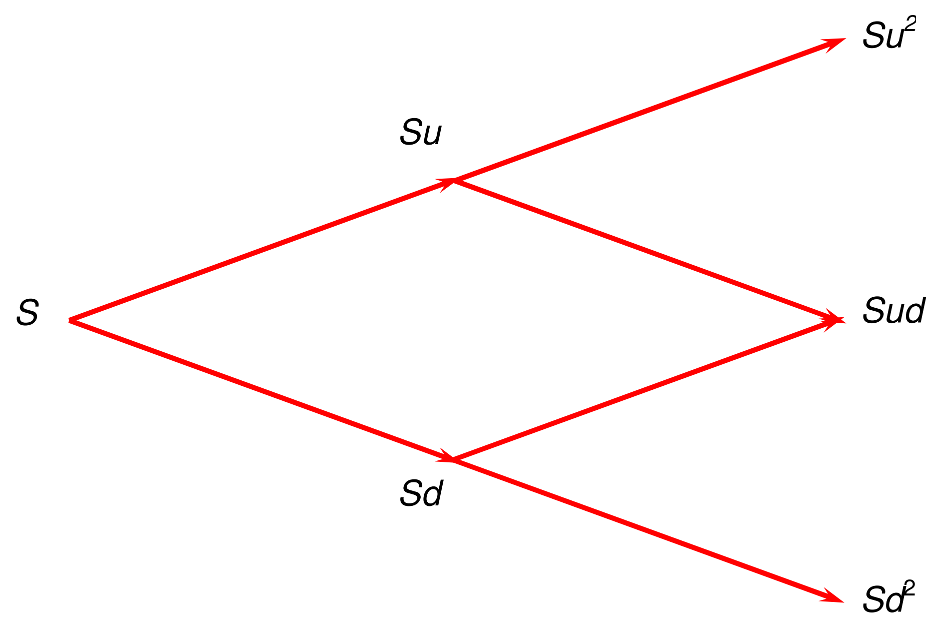
• Two main models	binomial lattices	discrete range of possible prices in each period
	Ito processes	continuous range

Binomial lattices

Binomial lattices are a version of binomial trees with recombining branches (or arches)

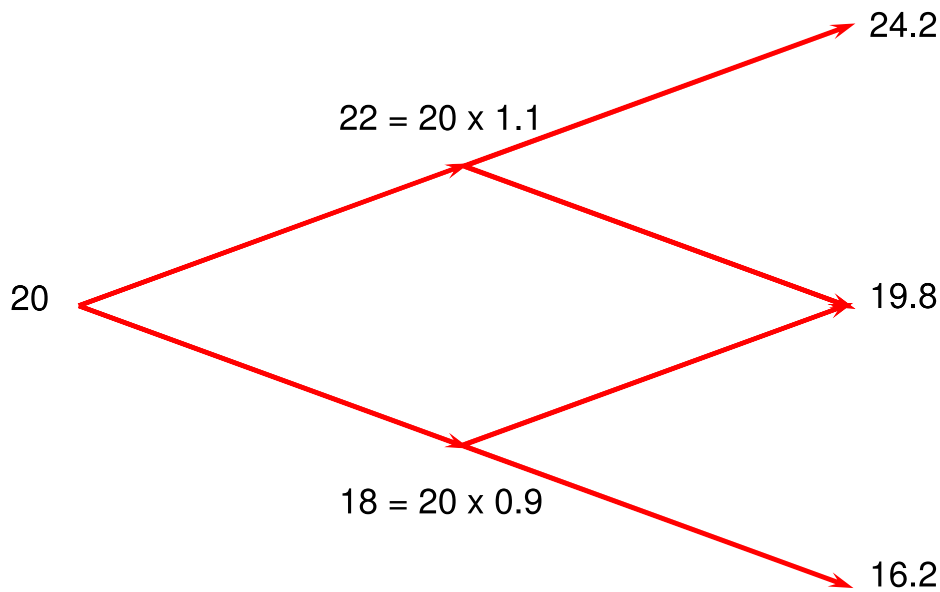
- The **multiplicative binomial lattice**
  - If $u(k) = u$ with probability p or d with probability $(1 - p)$ and with $u > 1$ and $d < 1$, the multiplicative model gives the binomial lattice
  - Means that the price can either go up by a factor $u > 1$ (total return > 1) or down by a factor $0 \leq d < 1$ (total return < 1) with probability p and $(1 - p)$, respectively
    - very useful to represent the behavior of asset prices
    - because the model is multiplicative, the price will never become negative...makes sense
    - we have a lattice because $ud = du$

    in situations when <up movement, down movement> ≠ <down movement, up movement> we have a tree with non recombining branches .... this is typical of decision trees where outcome depends on more than one decision variable (can you think of an example?...if curious, see Ch 5.3 for more details)



Two-period example:

$S = 20$
$u = 1.1$
$d = 0.9$



- To specify a model for the dynamic behavior of an actual asset, we need to pick values for u, d and p as 'realistically' as possible
  - What does 'realistically' mean here?
  - means that we should be able to reproduce the actual stochastic behavior of the asset as closely as possible
  - Since the price never takes negative values we can work with logs (more convenient)

  - Recall
    - $R = S_T/S_0$ growth
    - $r = ln(S_T/S_0)$ growth rate
  - Define
    - $v = E[ln(S_T/S_0)]$ expected growth rate
    - $\sigma^2 = var[ln(S_T/S_0)]$ variance of growth rate

- To match the stochastic properties of the actual stock price, we must make sure that the expected growth rate and variance of the stock in the binomial model match the actual ones

- In the binomial model, letting S(0) = 1:

$$E[ln(S_1/S_0)] = E[ln(S_1)] = p \ln u + (1 - p) \ln d$$

$$\begin{aligned} var[ln(S_1/S_0)] &= var[ln(S_1)] \\ &= E[ln(S_1)^2] - E[ln(S_1)]^2 \\ &= [p \ln u^2 + (1 - p) \ln d^2] - [p \ln u + (1 - p) \ln d]^2 \\ &= p(1 - p) (\ln u - \ln d)^2 \end{aligned}$$

  - At every node after the first the binomial process is identical, so these same mean and variance continue to apply

- Next, for any period of length Δt, impose that the mean and variances of the binomial model match the actual mean and variances

$$p \ln u + (1 - p) \ln d = v\Delta t$$
$$p(1 - p) (\ln u - \ln d)^2 = \sigma^2\Delta t$$

or

$$p\, U + (1 - p)\, D = v\Delta t$$
$$p(1 - p) (U - D)^2 = \sigma^2\Delta t$$

This is a system of 2 equations in 3 unknowns, U, D and p, so we have one extra degree of freedom. So, we might let $D = - U$ or

$$\ln u = - \ln d$$
$$\exp(\ln u) = \exp(- \ln d) = \exp(-1 \ln d) = \exp[\ln(d^{-1})]$$
$$u = 1/d$$

Then we can write

$$p\, U - (1 - p)\, U = 2pU - U = \quad (2p - 1)U = v\Delta t$$
$$p(1 - p) (U - D)^2 = \quad 4p(1 - p)U^2 = \sigma^2\Delta t$$

To solve, square the 1st equation and add to the second...you get:

$$U = \sqrt{\sigma^2 \Delta t + (v\Delta t)^2} = \ln u = -\ln d$$

and

$$p = \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2 / (v^2 \Delta t) + 1}}$$

- For small Δt, u, d and p can be approximated as follows:

$$p = \frac{1}{2} + \frac{1}{2}\frac{v}{\sigma}\sqrt{\Delta t}$$
$$u = e^{\sigma\sqrt{t}}$$
$$d = e^{-\sigma\sqrt{t}}$$

- With these values of u, d and p, the binomial model should match closely the actual stochastic behavior of the asset under consideration....as long as this can be described entirely by its mean and variance.
    - This is the case with normally distributed (log) prices

Example (A Volatile Stock)

Annualized data:

v = 15%
σ = 30%

What are the <u>weekly</u> u, d and p of the corresponding binomial model?

$$u = e^{\sigma\sqrt{t}} = e^{0.30\sqrt{52}} = 1.04248$$
$$d = 1/1.04248 = 0.95925$$
$$p = \frac{1}{2} + \frac{1}{2}\frac{0.15}{0.30}\sqrt{52} = 0.534669 \cong 53\%$$

- Risk-neutral dynamics:

$E^Q(S_{t+1}) = S_0u(q) + S_0d(1 - q) = S_0e^r$

$\Rightarrow \quad S_0uq + S_0d - S_0dq = S_0e^r$

$\Rightarrow \quad uq + d - dq = e^r$

$\Rightarrow \quad uq - dq = e^r - d$

$\Rightarrow \quad q(u - d) = e^r - d$

$$\Rightarrow \quad q = \frac{e^r - d}{u - d} \qquad\qquad 1 - q = 1 - \frac{e^r - d}{u - d} = \frac{u - d - e^r + d}{u - d} = \frac{u - e^r}{u - d}$$

This is the central result of the well known **Binomial Option Pricing Model** (BOPM)

See application to option pricing in **binomial.ppt**

Ito Processes

Using Ito processes we can extend the multiplicative model of asset prices to a continuous time setting

For example, a generalized Weiner process for d[ln $S(t)$] can be seen as the continuous time counterpart of the multiplicative model:

Discrete time multiplicative model for Δ[ln $S(t)$]		**Generalized Weiner process for d[ln $S(t)$]**
$\ln S(k+1) - \ln S(k) = w(k)$	$\rightarrow$	$d \ln S(t) = v\, dt + \sigma\, dz$
$w(k)$ iid normal		dz Weiner process v and σ constants

- d ln $S(t)$ is the INSTANTANEOUS continuously compounded growth rate of the asset price

  $E[d \ln S(t)] = v\, dt + \sigma\, E[\varepsilon(t)] \sqrt{dt} = v\, dt$

  $var[d \ln S(t)] = \sigma^2\, E[\varepsilon(t)^2] dt = \sigma^2\, dt$

- 'Solving' the generalized Wiener process for ln $S(t)$,

  $\ln S(t) = \ln S(0) + vt + \sigma\, z(t)$ **Geometric Brownian Motion**

  $\ln S(t) - \ln S(0) = \ln [S(t)/S(0)] = \ln (R)$ is the continuously compounded growth rate of the asset price *ab initio* (since an initial time 0)

  $E[\ln S(t)] = \ln S(0) + vt$ =>

  - ⇨ linear growth of the expected log
  - ⇨ $vt = E \ln [S(t)/S(0)]$

- $\ln S(t) \sim N(\ln S(0) + vt, \sigma^2 t)$

  - Geometric Brownian Motion is **log-normal**
  - just like the discrete time multiplicative process with normal disturbances $w(k)$)

- $S(t) = e^{\ln S(0) + vt + \sigma z(t)} = e^{\ln S(0)}\, e^{vt + \sigma z(t)} = S(0)\, e^{vt + \sigma z(t)}$

- But

$$E[S(t)] = S(0)E[e^{vt+\sigma z(t)}]$$
$$\neq S(0)e^{E[vt+\sigma z(t)]}$$
$$\neq S(0)\, e^{vt}$$

!!!!!!!!!!!!!!!!!!!!! (again, we need Ito's lemma to prove this)

Standard Ito Form for *S*(*t*)

- Recall the process for the *instantaneous continuously compounded* growth rate,

  $$d \ln S(t) = v \, dt + \sigma \, dz$$

- $\frac{dS(t)}{S(t)}$ can be seen instead as the *instantaneous* growth rate of the asset price

- If σ = 0, from ordinary calculus the process for $\frac{dS(t)}{S(t)}$ is

$$d \ln S(t) = \frac{dS(t)}{S(t)}$$

$$\Rightarrow vdt + \sigma dz = \frac{dS(t)}{S(t)}$$

$$\Rightarrow \frac{dS(t)}{S(t)} = vdt + \sigma dz$$

The process for the instantaneous growth rate could then be 'solved' for *S*(*t*)

However, when σ = 0, ordinary calculus breaks down (because Wiener processes are not differentiable w.r.t. time...recall previous discussion)

The appropriate Ito process for $\frac{dS(t)}{S(t)}$ is

$$\frac{dS(t)}{S(t)} = \left(v + \frac{1}{2}\sigma^2\right)dt + \sigma dz$$

$$= \mu dt + \sigma dz$$

This is an alternative representation of the geometric Brownian motion followed by *S*(*t*). So,

$$E\left[\frac{dS(t)}{S(t)}\right] = \mu dt$$

$$= v + \frac{1}{2}\sigma^2$$

$$\neq v = E[d \ln S(t)]$$

To prove this result, as for many other applications in options theory, we need **Ito's Lemma**

Ito's Lemma

Suppose that the random process x is defined by the Ito process ,

$$\mathrm{d}x(t) = a(x, t)\,\mathrm{d}t + b(x, t)\,\mathrm{d}z$$

Where z is a Wiener process

Suppose also that the process $y(t)$ is defined as

$$y(t) = F(x, t)$$

Then $y(t)$ satisfies the Ito process

$$dy(t) = \left(\frac{\partial F}{\partial x} a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b^2\right) dt + \frac{\partial F}{\partial x} b dz$$

Note 1: z is the same Wiener process as in the model for $x(t)$

Note 2: in ordinary calculus the term $\frac{1}{2}\frac{\partial^2 F}{\partial x^2} b^2$ would not be there

Proof:

$$
\begin{aligned}
y+\Delta y &= F(x,t)+\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial t}\Delta t \\
&= F(x,t)+\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\Delta x)^2 \\
&= F(x,t)+\frac{\partial F}{\partial x}(a\Delta t+b\Delta z)+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a\Delta t+b\Delta z)^2 \\
&= F(x,t)+\frac{\partial F}{\partial x}(a\Delta t+b\Delta z)+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}[a^2(\Delta t)^2+b^2(\Delta z)^2+2ab\Delta t\Delta z]
\end{aligned}
$$

........recall that $\Delta z=\varepsilon(t)\sqrt{\Delta t}$, so Δz is of order 0.5 in Δt, $(\Delta z)^2$ is of order 1, $\Delta t\Delta z$ is of order 1.5

So, the term $a^2(\Delta t)^2$ and $2ab\Delta t\Delta z$ can be dropped because they are of higher order than 1 in Δt, but $b^2(\Delta z)^2$ can't because it is of order 1

$$
\begin{aligned}
&= F(x,t)+\frac{\partial F}{\partial x}(a\Delta t+b\Delta z)+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2(\Delta z)^2 \\
&= F(x,t)+\frac{\partial F}{\partial x}a\Delta t+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2(\Delta z)^2+\frac{\partial F}{\partial x}b\Delta z
\end{aligned}
$$

Moreover, $\lim_{\Delta t\to 0}(\Delta z)^2=\lim_{\Delta t\to 0}\left[\varepsilon(t)\sqrt{\Delta t}\right]^2=\lim_{\Delta t\to 0}\varepsilon(t)^2\Delta t=E[\varepsilon(t)^2]\Delta t=\Delta t$, so

$$
\begin{aligned}
&= F(x,t)+\frac{\partial F}{\partial x}a\Delta t+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\Delta t+\frac{\partial F}{\partial x}b\Delta z \\
&= F(x,t)+\left(\frac{\partial F}{\partial x}a+\frac{\partial F}{\partial t}+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)\Delta t+\frac{\partial F}{\partial x}b\Delta z
\end{aligned}
$$

Application of Ito's Lemma

Proposition:

$S(t)$ is a geometric Brownian motion governed by the equivalent processes

$$d[\ln S(t)] = v\,dt + \sigma\,dz$$

or

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz$$

Corollary:

$$E\left[\frac{dS(t)}{S(t)}\right] = \mu dt$$

$$= \left(v + \frac{1}{2}\sigma^2\right)dt \qquad \text{(proof of earlier result)}$$

$$\neq vdt = E[d \ln S(t)]$$

Proof (using Ito's Lemma):

Start from comparing

$$dx = a(x,t)dt + b(x,t)dz$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dz$$

so, let

$$dx = dS(t)$$
$$a(x,t) = \mu S(t)$$
$$b(x,t) = \sigma S(t)$$

To apply Ito's Lemma, let $F(x, t) = F[S(t)] = \ln S(t)$; so, the process governing $\ln S(t)$ is the following

$$dy(t) = \left(\frac{\partial F}{\partial x} a + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} b^2 \right) dt + \frac{\partial F}{\partial x} b dz \Rightarrow$$

$$d \ln S(t) = \left(\frac{1}{S(t)} a - \frac{1}{2} \frac{1}{S(t)^2} b^2 \right) dt + \frac{1}{S(t)} b dz$$

$$= \left(\frac{1}{S(t)} \mu S(t) - \frac{1}{2} \frac{1}{S(t)^2} \sigma^2 S(t)^2 \right) dt + \frac{1}{S(t)} \sigma \; S(t) \; dz$$

$$= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz$$

$$= v dt + \sigma dz$$

To simulate, 'discretize' the Ito process for the log-price:

$$d[\ln S(t)] = v\, dt + \sigma\, dz$$

$$\Rightarrow \quad \Delta\, [\ln S(t)] = v\, \Delta t + \sigma\, \Delta z$$

$$\Rightarrow \quad \ln S(t_{k+1}) - \ln S(t_k) = v\, \Delta t + \sigma\, \varepsilon(t_k)\sqrt{\Delta t}$$

$$\Rightarrow \quad S(t_{k+1}) = = S(t_k) e^{v\Delta t + \sigma\varepsilon(t_k)\sqrt{\Delta t}}$$

See notes on MC simulations

Relations for Brownian Motion

- Given the geometric Brownian motion $S(t)$

  d[ln $S(t)$] = v dt + σ dz

  $$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz$$

  $$E\left[\frac{\ln S(t)}{\ln S(0)}\right] = vt$$

  $$stdev\left[\frac{\ln S(t)}{\ln S(0)}\right] = \sigma\sqrt{t}$$

  $$E\left[\frac{S(t)}{S(0)}\right] = e^{\mu t}$$

  $$stdev\left[\frac{S(t)}{S(0)}\right] = e^{\mu t}\sqrt{e^{\sigma^2 t} - 1}$$

- Risk-neutral dynamics:

  $E^Q(S_{t+1}) = S_0 e^r$

  $$\Rightarrow E\left[\frac{S(t)}{S(0)}\right] = e^{\mu t} = e^{rt}$$

  $$\Rightarrow \mu = r$$

  Girsanov Theorem ensures that, when changing the drift rate to switch to the risk neutral measure, volatility stays the same.