Linear Pricing and RNV

No Arbitrage Pricing

References: Neftci Ch 11 (but this is a simplified treatment)

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Preliminary notions

- Security: random payoff d with price P
- The payoff, e.g. stock dividend, is revealed and obtained at the end of the period

Arbitrage

- Type A arbitrage: when an investment produces an immediate positive reward with no future payoff
  - e.g. like investing in a security with negative price and zero payoff
- Type B arbitrage: when an investment has non-positive cost but has a positive probability of a positive payoff and no probability of a negative payoff
  - e.g. when you pay nothing (or a negative amount) and you have the chance of getting something

Linear Pricing

- Assumption: the most basic form of arbitrage (Type A) is not possible
- Linear pricing (law of one price) follows
- Linear pricing says that for every real numbers a and b

$$P(ad_1 + bd_2) = P(ad_1) + P(bd_2)$$

Example:

- Consider security d with price P. Suppose that security $2d$ costs $P' < 2P$
- Is the no Type A arbitrage assumption satisfied?
- You could buy security $2d$ at $P' < 2P$, break it up in two d securities and sell them on for P each

1. Total cost is $P' < 2P$

2. Total revenue is $2P$

3. Arbitrage profit is $2P - P' > 0$

No Arbitrage

If neither Type A nor Type B arbitrage is possible, we say that there is **no arbitrage** possibility

Relative Pricing Preliminaries (Finite State Models)

- Suppose that the outcome of an investment can be described using S (finite) possible **states** of the world
- A **security** is a set of payoffs, one payoff each possible state
  - hence securities are vectors $d = <d^1, d^2 \ldots d^S>$
- An **elementary security** has a **unit** payoff in one state only, e.g. for state s
  - $d_s = <0, 0, ..1, 0,.. 0>$
- If the elementary state s securities exists, its price is denoted as ψ_s

Relative Pricing
(Elementary State Pricing)

- If the elementary securities exists, pricing any other securities is trivial, ...i.e. by linear pricing:

$$P = \sum_{s=1}^{S} d^{s} \psi_{s}$$

- If elementary state price securities do not exists it is might be possible to construct them artificially from traded securities prices, as long as the market is <u>**complete**</u>

- In a complete market there are at least as many independent security prices as there are states

Example

- Only securities *x* and *y* are traded:
  - $x = \langle 2, 1 \rangle$ $y = \langle 1, -1 \rangle$
- Elementary security <1, 0> replication

$$s = 1 \quad n_x\, 2 + n_y\, 1 = 1$$
$$s = 2 \quad n_x\, 1 + n_y\, (-1) = 0$$

- From 2nd equation:

$$n_x = n_y$$

- So, substituting into 1st equation

$$3n_y = 1$$
or
$$n_y = 1/3 \qquad \text{and} \qquad n_x = 1/3$$

Now, if we know the price of x and y, we can derive the price of the elementary security as a linear combination

When we have the price of the 2 elementary securities corresponding to the 2 states of the world, we can price any security with payoffs in only those two states

Positive State Prices

- The elementary security prices should be positive or you could bundle together a security with negative price (Type B arbitrage)
- **Positive State Prices Theorem**: A set of positive security prices exists *iff* there is no arbitrage opportunity

Risk Neutral Pricing

- Suppose there are S positive state prices, then the price of any security with payoffs in the S states is:

$$P = \sum_{s=1}^{S} d^s \psi_s$$

- Let's normalize the state prices s.t. they sum to 1:

$$P = \psi_0 \sum_{s=1}^{S} \frac{\psi_s}{\psi_0} d^s = \psi_0 \sum_{s=1}^{S} q_s d^s$$

Where

$$\psi_0 = \sum_{s=1}^{S} \psi_s \qquad q_s = \frac{\psi_s}{\psi_0}$$

Risk Neutral Probabilities

- The normalizes state prices q_s can be seen as artificial probabilities (as they sum to 1)

- Then the pricing formula can be rewritten using an expectation defined over these artificial probabilities

$$P = \psi_0 \sum_{s=1}^{S} q_s d^s = \psi_0 \hat{E}(d)$$

Risk Neutral Probabilities

- By construction, ψ_0 is the price of the security that pays 1 in every state, i.e. the risk free security

- Thus

$$\psi_0 = \frac{1}{1+r}$$

- And the pricing formula becomes:

$$P = \frac{1}{1+r}\hat{E}(d) \qquad q_s = \frac{\psi_s}{\psi_0} = (1+r)\psi_s$$

- This is a **risk neutral** pricing formula, under the artificial **risk neutral** probabilities q_s

Little Exercise

Consider the following alternative propositions:

Proposition I:	You give me Eur 1. I flip a coin. If it is heads, you are paid Eur 2, if it is tails you are paid Eur 0. You might do so repeatedly. Each time, you might also give me multiples of Eur 1 and the payoffs scale accordingly.
Proposition II:	You might keep your money
Proposition III:	I flip a coin 3 times. If it is heads twice you are paid Eur 8, otherwise you are paid Eur 0.
Proposition IV:	I flip a coin 2 times. You are paid Eur 2 unless it is heads twice. In this case you are paid Eur 6.

Questions:

i. How much would you pay, at most, to take the bet defined by Proposition (III)?

ii. How much would you pay, at most, to take the bet defined by Proposition (IV)?

Little Exercise Solution

i. The price of state 'head' is 0.5 (i.e. price/payoff, ½). The risk-free return (Proposition II) is 1. Thus, the risk-neutral probability of head is 0.5.

You can get 8 with following sequences:

H, H, H prob: $0.5^3 = 0.125$
T, H, H prob.: $0.5 \times 0.5^2 = 0.125$
H, T, H prob.: $0.5^2 \times 0.5 = 0.125$
H, H, T prob.: $0.5^2 \times 0.5 = 0.125$

The risk-neutral valuation of pay-offs from (III) is therefore

$$0.125 \times 4 \times 8 = 4$$

ii. Proposition (IV) can be seen as a derivative of the first 2 propositions. The risk neutral probability of head is 0.5.

Thus,

a. The risk neutral probability of 'it is head twice' is 0.25 and therefore the risk neutral valuation of the Eur 6 payoff is Eur 6/4 (= Eur 6 × 0.25);
b. The risk neutral probability of 'it is not head twice' is 0.75 and therefore the risk neutral valuation of the 2 Eur payoff is Eur 6/4 (= Eur 2 × 0.75) too.

Hence, the proposition is worth Eur 12/4 (= 6/4 + 6/4), or Eur 3.

Little Exercise Again

A ‘bookie’ is quoting the following odds:

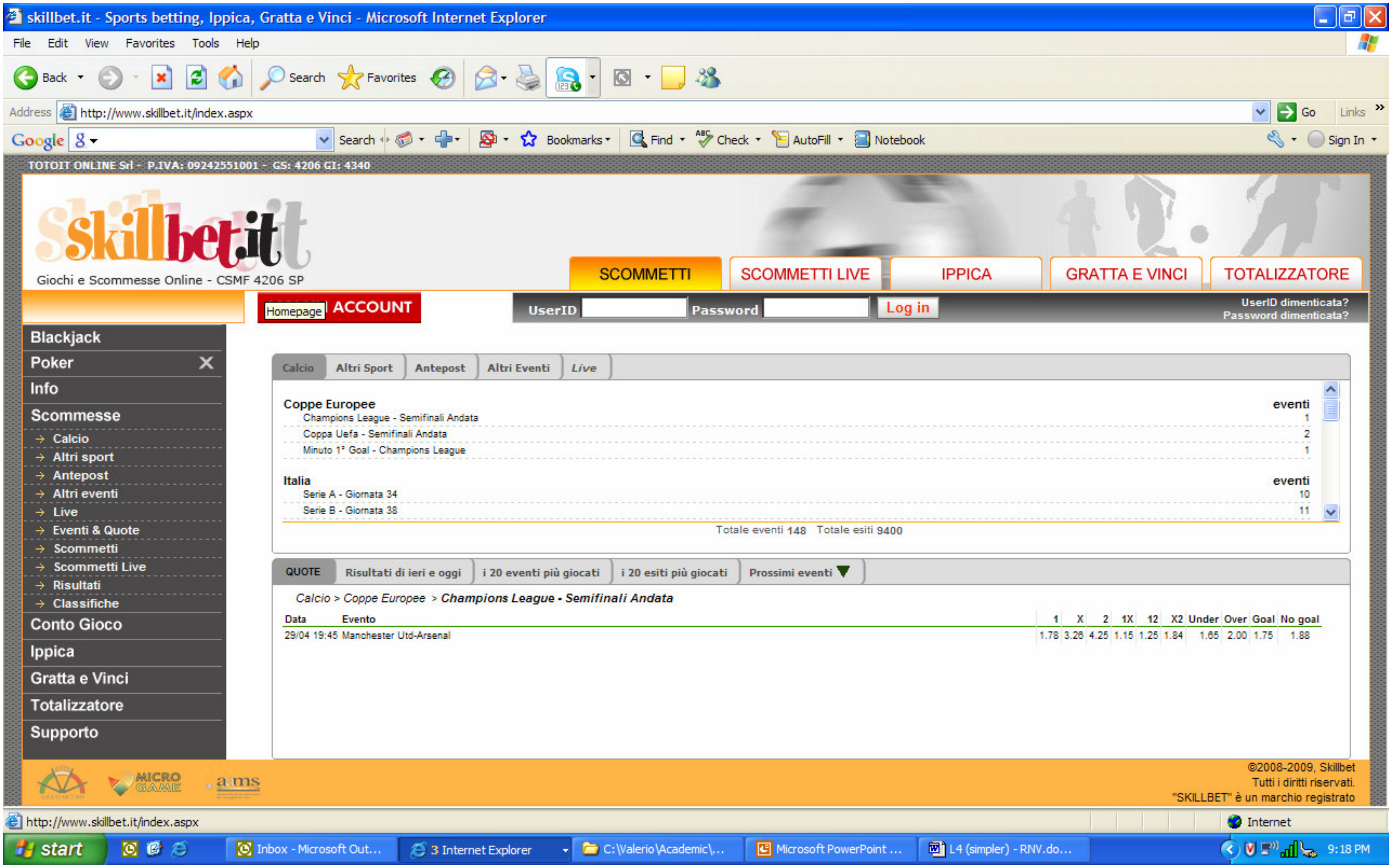
Bet 1: Juventus wins UEFA Cup	2 to 1
Bet 2: Milan wins Campionato	2 to 1
Bet 3: Lecce stays in A	2 to 1

If you were the ‘bookie’ how would you set the odds for the following bets, if your aim was to break even?

Bet 4: any two events considered by Bets 1, 2 and 3 occurr
Bet 5: the events considered by both Bet 1 or 2 occurr

What about the odds if your aim was a 50% return?

What about the odds quoted by he ‘bookie’ on the screen next page? What is the bookie target risk-free return?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Italia
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Serie A - Giornata 34 10
Serie B - Giornata 38 11
Totale eventi 148 Totale esiti 9400
QUOTE
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Prossimi eventi
Calcio > Coppe Europee > Champions League - Semifinali Andata
Data Evento 1 X 2 1X 12 X2 Under Over Goal No goal
29/04 19:45 Manchester Utd-Arsenal 1.78 3.26 4.25 1.15 1.25 1.84 1.65 2.00 1.75 1.88
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Monte Carlo Simulation and Options RNV

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoff
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option