# Financial Engineering

Introduction

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# What is financial engineering?

- Definition 1:

  Combining or carving up existing instruments to create new financial products. (Campbell R. Harvey's Hypertextual Financial Glossary, http://www.duke.edu/~charvey/Classes/wpg/glossary.htm)

- Definition 2:

  The application of mathematical tools commonly used in physics and engineering to financial problems, especially the pricing and hedging of derivative instruments (from Neil D. Pearson, Harry A. Brandt Professor of Financial Markets and Options, UIUC)

- Definition 3:

  The use of financial instruments such as forwards, futures, swaps, options, and related products to restructure or rearrange cash flows in order to achieve particular financial goals, particularly the management of financial risk.

**We shall adopt mainly Definition 1 and 3**

# Essentials

- So, financial engineering is essentially about **REPLICATION**, that is replicating complex payoffs using elementary ones
- Payoffs are often, but not necessarily, embedded in securities
- Securities and other financial and non financial contracts and assets can be seen as **contingent claims** to future payoffs, i.e. claims contingent on the realization of certain conditions
- **Linear/non linear payoffs**: the payoff depends linearly or non-linearly on another payoff

# Preliminaries

- Textbook notation:

  Suppose we are at time $t_i = t_0$ and we subdivide time in periods of length

  $\Delta t_{i+1} \equiv t_{i+1} - t_i \equiv \delta$

  Thus, $\delta$ represents the number of time units (e.g., a year) in a period (e.g., $\delta = 1$ for a 1-year period, $\delta = ½$ for a 6-month period, etc.)

- If a security represents a claim to a payoff in $t_i$ we say that its maturity date is $T_i \equiv t_i$

- The above is the textbook (Neftci) notation, and I wil try and stick to it when possible.

- When this is too painful, however, I will use my own simplified notation

- Simplified notation:

  We will let $t = t_i$ and therefore

  $\Delta t_{i+1} \equiv \Delta t \equiv (t + 1) - t$

  This is also more or less the notation of the other possible reference (Cuthbertson & Nitzche)

- I will also often assume (either explicitly or implicitly), $\delta = 1$

## Arbitrage

- Type A arbitrage: when an investment produces an immediate positive reward with no future payoff
  - e.g. like investing in a security with negative price and zero payoff

- Type B arbitrage: when an investment has non-positive cost but has a positive probability of a positive payoff and no probability of a negative payoff
  - e.g. when you pay nothing (or a negative amount) and you have the chance of getting something

## Linear Pricing

- Assumption: the most basic form of arbitrage (Type A) is not possible
- Linear pricing (LOOP) follows
- Linear pricing says that, for every real numbers $a$ and $b$ and payoffs $z_1$ and $z_2$,

$$P(az_1 + bz_2) = P(az_1) + P(bz_2)$$

# Example:

- Consider security with payoff $z_1 = d$ and price $P$. Suppose that security with payoff $z_2 = 2d$ costs $P' < 2P$
- Is the no Type A arbitrage assumption satisfied?
- You could buy security with $2d$ payoff at $P' < 2P$, break it up in two $d$-payoff securities and sell them on for $P$ each

1. Total cost is $P' < 2P$
2. Total revenue is $2P$
3. Arbitrage profit is $2P - P' > 0$

## No Arbitrage

If neither Type A nor Type B arbitrage is possible, we say that there is **no arbitrage** possibility

# A simple economy

- In the simple economy in which we live, there are only three liquid assets

- The first asset is a **deposit**, where one can place an amount $B(t_0)$ of cash, earn an interest $L(t_i)$ per unit of time (e.g., a year) and, after n periods of length $\delta$ (the tenor), obtain $B(t_n)$,

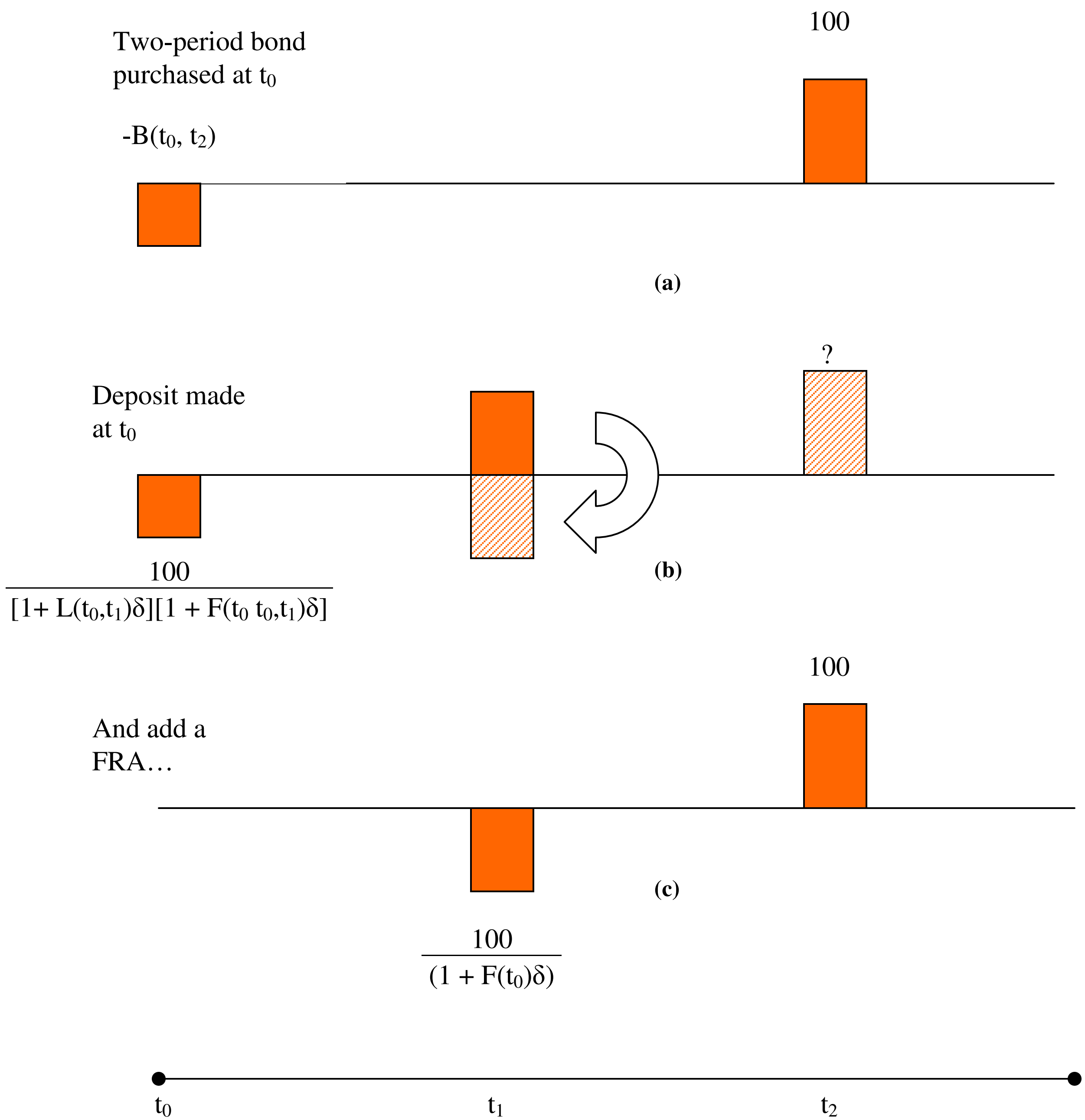
$$B(t_n) = B(t_0)(1+L(t_0)\delta)(1+L(t_1)\delta)\ldots(1+L(t_n)\delta)$$

- The second asset is a "zero coupon" **bond** that matures in <u>two</u> periods, with price $B(t_0,T_2)$ and face value $B(t_2,T_2) = 100$

- The third security is a forward rate agreement (**FRA**) that specifies at $t_0$ the rate $F(t_0)$ offered on a one-period deposit to be made in $t_1$, regardless of what $L(t_1)$ turns out to be

# A (static) replication problem

- Suppose we wish to create the two-period bond synthetically
- Easy, just combine a one-period deposit with the FRA


100
Two-period bond
purchased at $t_0$
$-B(t_0, t_2)$
(a)
?
Deposit made
at $t_0$
$\frac{100}{[1+ L(t_0,t_1)\delta][1 + F(t_0\ t_0,t_1)\delta]}$
(b)
100
And add a
FRA…
(c)
$\frac{100}{(1 + F(t_0)\delta)}$
$t_0$
$t_1$
$t_2$

# Question

- What if nobody trades FRAs?
- Then, we'd have a problem… (static) replication as in the previous example would be impossible
- Would any replication at all be possible?

# Hedging/Immunization

- Let's try and replicate the 2-period bond with a 1-period zero-coupon bond or discount loan, corresponding to making a 1-period deposit that matures at 100, i.e.

  $B(t_i, T_{i+1}) = 100$

- For simplicity assume the period is one year

- Assume the term structure is "flat", i.e. the yield to maturity $y(t_i, T_{i+k})$ is the same across maturities $T_{i+k}$, i.e. $y(t_i, T_{i+k}) = y(t_i)$, and the current value of the yield is 5% p.a., i.e. $y(t_0) = 5\%$

  $B(t_0, T_1) = 100(1 + y)^{-1}$ this is the PV/price at $t_0$ of the 1-year loan

  $B(t_0, T_2) = 100(1 + y)^{-2}$ this is the PV/price at $t_0$ of the 2-year bond

- Then,

  $dB(t_0, T_1)/dy = -100(1+y)^{-2} = -90.7$

  $dB(t_0, T_2)/dy = -200(1+y)^{-3} = -172.8$

- We want to form an imaginary portfolio of the loan and the 2-period bond and see if we can "immunize" it from changes in value between time $t_0$ and time $t_{0+\varepsilon} = t_\varepsilon$,

$$dV_\varepsilon \equiv d[\omega_{loan,0}\, B(t_\varepsilon,T_1) + \omega_{bond,0}\, B(t_\varepsilon,T_2)] = 0$$

- So,

$$\omega_{loan,0}\, dB(t_0,T_1) + \omega_{bond,0}\, dB(t_0,T_2) = 0$$

- Or,

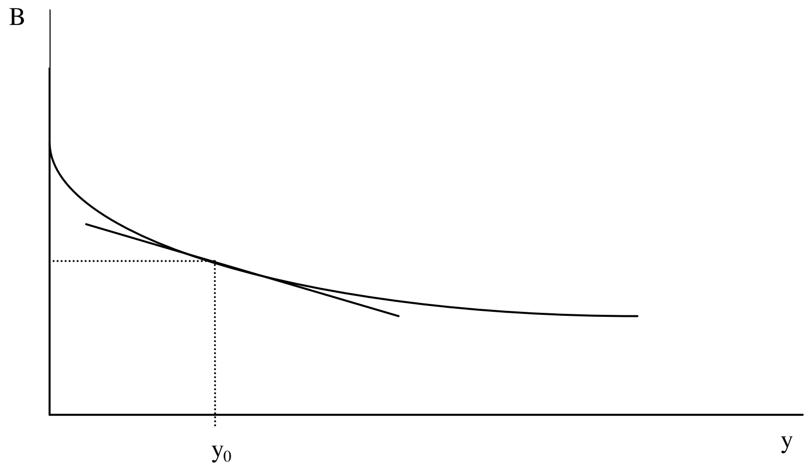
$$- \omega_{loan,0}\, 90.7\, dy - \omega_{bond,0}\, 172.8\, dy = 0$$

- Dividing through by dy and solving for $\omega_{loan,0}$, we find that we need to 'short' 1.9 units of the loan for each unit of the bond held

$$\omega_{loan,0} = - \omega_{bond,0}\, 172.8/90.7 = - 1.9\omega_{bond,0}$$

# Immunization/Hedging vs. Replication

- Have we achieved our aim of replicating the bond?
- The problem is that $dB(t_0,T_1)/dy$ and $dB(t_0,T_2)/dy$ change when y changes (see graph below)
- So we need to continually re-adjust the hedge, by either **injecting** or **withdrawing** cash
- This creates reinvestment risk and we cannot claim we have managed to replicate the payoff of interest


B
y
y0

# Dynamic replication

- What if we used a portfolio of assets to replicate the bond and switched excess-cash between the constituent assets so as to avoid injections and withdrawals?
- This is the idea behind dynamic replication, i.e. form **dynamically rebalanced** replicating portfolios
- Can be used also to replicate other securities, e.g. dynamic replication of options on stocks with the underlying stock
- We'll see this later

# Summary

- **Static replication**: when you combine payoffs in fixed proportions
- **Dynamic replication**: when payoffs need to be combined in changing proportions in order to replicate the desired payoff
- Static replication is a special case of dynamic replication with fixed 'weights'
- Can be used for hedging and, under LOOP, pricing