DUBLIN CITY UNIVERSITY

SEMESTER TWO REPEAT EXAMINATION 2007-2008

MODULE:	MS408/MS408M/SHSAX/SHSAO Probability & Finance II
COURSE:	M. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences B. Sc. in Financial and Actuarial Mathematics Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. Hurley Prof. B. Hanzon
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) State and prove Jensen's inequality.

[10 marks]

(b) An investor spreads his funds between a cash account that pays a constant rate of interest r and a stock, the unit price of which obeys

$$dS_t = \alpha S_t dt + \sigma S_t dB_t.$$

It can be shown that, if π_t denotes the proportion of funds held in stock at time t, the investor's wealth is

$$V_t = V_0 e^{\int_0^t [r+(\alpha-r)\pi_s - \frac{\sigma^2}{2}\pi_s^2]ds + \sigma\int_0^t \pi_s dB_s}.$$

Assuming this, find the strategy which maximises $E[V_T^\gamma]$ where $0 < \gamma < 1$.

[23 marks]

QUESTION 2

(a) Define standard Brownian motion.

[8 marks]

(b) Let the random variable L be the length of the path of standard Brownian motion $B_t, 0 \leq t \leq 1$. Establish the inequality

$$L \geq L_n \equiv \sum_{j=0}^{2^n-1} |B_{\frac{j+1}{2^n}} - B_{\frac{j}{2^n}}| \quad \forall n \in \mathbb{N}.$$

[6 marks]

(c) Prove that

$$\mathbb{E}[L_n] = \mathbb{E}[|B_1|]2^{\frac{n}{2}}, \quad \mathbb{D}[L_n] = \mathbb{D}[|B_1|].$$

[9 marks]

(d) Deduce from the above that

$$\forall \varepsilon > 0, \quad \mathbb{P}[L_n < (\mathbb{E}[|B_1|] - \varepsilon)2^{\frac{n}{2}} \quad i.o.] = 0$$

and thus prove that $L = \infty$ a.s..

[10 marks]

QUESTION 3

(a) Explain what is meant by a simple integrand Y_t adapted to the natural filtration of the standard Brownian motion B_t. Define the Itô integral

$$I_t[Y.] = \int_0^t Y_s dB_s$$

for such integrands.

[6 marks]

(b) Prove that $I_t[Y.]$ as defined above is a martingale.

[10 marks]

(c) Prove that $I_t[Y.]$ has continuous sample paths.

[8 marks]

(d) State a third property of $I_t[Y.]$ and indicate briefly how it can be used to extend the definition of the Itô integral to a suitable class of integrands $Y.$.

[9 marks]

QUESTION 4

Consider two currencies with constant interest rate r_a and r_b respectively. Suppose that the exchange rate (defined as the value in currency b of one unit of currency a) obeys the stochastic differential equation

$$dX_t = \alpha X_t dt + \sigma X_t dB_t$$

where B_t is standard Brownian motion.

(a) Identify the following from the point of view of an investor whose home currency is b: the riskless asset, the basic risky tradable asset, the risk-neutral measure.

[10 marks]

(b) Hence price a call which gives you the option to buy at time T one unit of currency a for K units of currency b.

[13 marks]

(c) Explain briefly what changes if the problem is seen from the point of view of an investor whose home currency is a.

[10 marks]