DUBLIN CITY UNIVERSITY

SEMESTER TWO EXAMINATION 2007-2008

MODULE:	MS408/MS408M/SHSAX/SHSAO Probability & Finance II
COURSE:	M. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences B. Sc. in Financial and Actuarial Mathematics Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. Hurley Prof. B. Hanzon
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) State and prove the two Borel-Cantelli lemmas.

[25 marks]

(b) Define almost sure convergence for sequences of random variables and prove that $X_n \to X$ a.s. if and only if

$$\forall \varepsilon > 0, \mathbb{P}[|X_n - X| > \varepsilon \quad i.o.] = 0$$

[8 marks]

QUESTION 2

(a) When is a probability space said to be complete? Discuss briefly the role of completion in the theory of continuous-time stochastic processes.

[6 marks]

(b) Define standard Brownian motion $B_t, t \geq 0$.

[4 marks]

(c) Prove that $\forall t_0, \quad \mathbb{P}[t \mapsto B_t \text{ is differentiable at } t_0] = 0$.

[19 marks]

(d) Does the above prove that $\mathbb{P}[t \mapsto B_t \text{ is differentiable at some } t_0] = 0$? Explain why.

[4 marks]

QUESTION 3

Let $T = \inf\{t : B_t = a \quad \text{or} \quad B_t = b\}$ be the first exit time of Standard Brownian Motion from the band (a, b) where $a < 0 < b$.
(a) Use the martingale B_t to prove that $\mathbb{P}[B_T = a] = \frac{b}{b-a}$.

[8 marks]

(b) Use the martingale $B_t^2 - t$ to prove that $\mathbb{E}[T] = -ab$.

[8 marks]

(c) Explain why, in the symmetric case ($b = -a$) the random variables T and B_T are independent.

[4 marks]

(d) In the symmetric case, use the martingale $e^{\lambda B_t - \lambda^2 \frac{t}{2}}$ to prove that

$$\mathbb{E}[e^{-\alpha T}] = \frac{1}{\cosh b\sqrt{2\alpha}}.$$

[13 marks]

QUESTION 4

(a) State Girsanov's theorem for the Brownian motion with drift $W_t = B_t + \mu t$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

[10 marks]

(b) Use Girsanov's theorem and the identity $(S_T - K)^+ = (S_T - K)I_A$ where A is the event $\{S_T \geq K\}$ to prove the Black-Scholes formula for a call option on a stock in the standard model.

[23 marks]