DUBLIN CITY UNIVERSITY

SEMESTER TWO REPEAT EXAMINATION 2005

MODULE:	MS408,MS408M Probability & Finance II
COURSE:	M. Sc. in Financial & Industrial Mathematics B. Sc in Mathematical Sciences B. Sc. in Financial & Actuarial Mathematics Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. M. H. A. Davis Prof. T. Hurley Mr. P. Cooper
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks.
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) State the two Borel-Cantelli lemmas and prove one of them.

[11 marks]

(b) Let X_1, X_2, ...X_n, ... be a sequence of independent identically distributed random variables with $\mathbb{E}[X_n] = \mu$, $\mathbb{D}[X_n] = \sigma^2$; consider

$$Y_n = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu).$$

(i) Calculate $\mathbb{E}[Y_n]$, $\mathbb{E}[Y_n^2]$, and thus prove that $Y_n \to 0$ in L^2 as $n \to \infty$.

[4 marks]

(ii) Use the above result to prove that the subsequence Y_{n^2} converges to 0 almost surely as $n \to \infty$.

[6 marks]

(iii) Prove the strong law of large numbers; you may find it useful to consider the sequence of random variables $Z_n = Y_n - \frac{p_n^2}{n} Y_{p_n^2}$ where p_n is the integer part of $\sqrt{n}$.

[12 marks]

QUESTION 2

(a) Define **standard Brownian motion**.

[8 marks]

(b) Let the random variable L be the length of the path of standard Brownian motion B_t, $0 \le t \le 1$. Establish the inequality

$$L \ge L_n \equiv \sum_{j=0}^{2^n-1} |B_{\frac{j+1}{2^n}} - B_{\frac{j}{2^n}}| \quad \forall n \in \mathbb{N}.$$

[6 marks]

(c) Prove that

$$\mathbb{E}[L_n] = \mathbb{E}[|B_1|]\, 2^{\frac{n}{2}}, \quad \mathbb{D}[L_n] = \mathbb{D}[|B_1|].$$

[9 marks]

(d) Deduce from the above that

$$\forall \varepsilon > 0, \quad \mathbb{P}[L_n < (\mathbb{E}[|B_1|] - \varepsilon)\, 2^{\frac{n}{2}} \quad i.o.] = 0$$

and thus prove that $L = \infty \quad a.s.$.

[10 marks]

QUESTION 3

(a) State Itô's lemma for $f(t, B_t)$, and use it to prove that $B_t^4 - 6t\, B_t^2 + 3t^2$ is a martingale with respect to the natural filtration of the standard Brownian motion B_t.

[11 marks]

(b) Let X_t and X'_t be Itô processes of the form

$$dX_t = Y_t\, dB_t + Z_t\, dt$$

$$dX'_t = Y'_t\, dB_t + Z'_t\, dt.$$

Obtain a formula for $d(X_t X'_t)$.

[10 marks]

(c) Use the above to prove the **uniqueness** of the solution of the stochastic differential equation

$$dS_t = \sigma S_t\, dB_t + \alpha S_t dt.$$

[12 marks]

QUESTION 4

(a) Calculate explicity the hedging strategy $\phi_t = \frac{\partial C_t}{\partial S_t}$ for a European call option in the Black-Scholes model. You may assume the Black-Scholes formula

$$C_t = S_t G(d_1) - K e^{r(t-T)} G(d_2)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2} \right) (T-t) \right), \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

[13 marks]

(b) Consider the following variant of the standard Black-Scholes model: the underlying stock pays continuously a fraction Δ of its value as dividend. Give an argument supporting the equation

$$dS_t = (\alpha - \Delta) S_t \, dt + \sigma S_t \, dB_t$$

for the value of the 'naked' stock (the ownership of which does not entitle you to receiving dividends). Form a tradable asset $\hat{S}_t$ related to S_t, and obtain an equation for $\hat{S}_t$.

[6 marks]

(i) Show how the standard Black-Scholes formula of (a) has to be modified to take the dividend payments into account. It is suggested that you translate the payoff $H = (S_T - K)^+$ in terms of the tradable asset $\hat{S}_t$.

[14 marks]