DUBLIN CITY UNIVERSITY

SEMESTER TWO EXAMINATION 2004 – 2005

MODULE:	MS408,MS408M Probability & Finance II
COURSE:	M. Sc. in Financial & Industrial Mathematics B. Sc in Mathematical Sciences B. Sc. in Financial & Actuarial Mathematics Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. M. H. A. Davis Prof. T. Hurley Mr. P. Cooper
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks.
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) Define the events $\{A_n \text{ i.o. }\}$ and $\{A_n \, ev.\}$. State and prove any relationship that may exist between these two events.

[8 marks]

(b) Define what it means for a sequence of random variables to converge

(i) almost surely

(ii) in L^2

(iii) in probability

(iv) in distribution.

[8 marks]

(c) Let $X_1, X_2, ..., X_n, ...$ be a sequence of independent two-valued random variables with the following probability distribution:

$$\mathbb{P}[X_n = \alpha_n] = p_n, \quad \mathbb{P}\,[X_n = 0] = 1 - p_n.$$

Assuming that $\alpha_n > 0$ and $\alpha_{n+1} \geq \alpha_n$, state necessary and sufficient conditions on α_n and p_n for X_n to tend to zero in the sense of each of the four definitions above. Provide detailed justifications for your answers.

[9 marks]

(d) Use the above to construct examples which demonstrate that the four modes of convergence listed in (b) are not equivalent to each other.

[8 marks]

QUESTION 2

The **reflection principle** states that the probability distribution of standard Brownian motion is unchanged if the sign of its increments is reversed after the **hitting time**: $T_b = \inf\{t : B_t = b\}$.

Let $\bar{G}$ denote the complementary standard Gaussian probability distribution function:

$$\bar{G}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-y^2}{2}} dy.$$

[9 marks]

Establish the formulas a), b), c) below:

(a)

$$\mathbb{P}[B_t \le a, \max_{0\le s\le t} B_s \ge b] = \bar{G}\left(\frac{2b-a}{\sqrt{t}}\right) \text{ when } a \le b, b > 0,$$

(b)

$$\mathbb{P}[\max_{0\le s\le t} B_s \ge b] = \begin{cases} 2\bar{G}\left(\frac{b}{\sqrt{t}}\right) & \text{if } b > 0, \\ 1 & b \le 0. \end{cases}$$

[8 marks]

(c)

$$\mathbb{P}[B_t \le a, \max_{0\le s\le t} B_s \ge b] = 2\bar{G}\left(\frac{b}{\sqrt{t}}\right) - \bar{G}\left(\frac{a}{\sqrt{t}}\right) \text{ when } a > b > 0.$$

[8 marks]

(d) Calculate the probability density of T_b.

[8 marks]

QUESTION 3

(a) Define the following terms for continuous-time stochastic processes:-

(i) filtration

(ii) adapted process

(iii) martingale

(iv) stopping time.

[8 marks]

(b) Let $\{B_t, t \ge 0\}$ be standard ö Brownian motion. Prove that $B_t^2 - t$ and $e^{\lambda B_t - \lambda^2 \frac{t}{2}}$ are martingales with respect to the natural filtration of B_t.

[9 marks]

(c) Construct a martingale based on B_t^3.

[6 marks]

(d) State Itô's lemma and use it to solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma S_t dB_t.$$

[10 marks]

QUESTION 4

(a) A stock has price

$$S_t = S_0 e^{\sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t}$$

where W_t is standard Brownian motion under the risk-neutral measure $\mathbb{P}^*$. Use the optional stopping theorem to prove the formula $\mathbb{E}^* \left[e^{-r\tau_a}\right] = \left(\frac{a}{S_0}\right)^{\frac{2r}{\sigma^2}}$ when $S_0 > a$, where τ_a is the first hitting time of a by S_t.

[18 marks]

(b) Check that if $S_0 > a$

$$\mathbb{E}^*[e^{-r\tau_a}\,(K - S_{\tau_a})^+] = (K - a)^+ \quad \mathbb{E}[e^{-r\tau_a}].$$

Hence, obtain a formula for the value of the perpetual American put when $S_0 > K$; you may assume that the optimal exercise time is a hitting time of S_t.

[15 marks]