DUBLIN CITY UNIVERSITY

SEMESTER TWO REPEAT EXAMINATION 2005-2006

MODULE:	MS407/MS407M/SHSAX/SHSAO Probability & Finance 1
COURSE:	M. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences B. Sc. in Financial and Actuarial Mathematics Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. Hurley Prof. B. Hanzon
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	None

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QUESTION 1

(a) Define the following: probability space, random variable; state the axioms of probability theory.

[11 marks]

(b) Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $\mu_X = \mathbb{P} o X^{-1}$ is a probability measure on $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra.

[11 marks]

(c) Let $X_1, X_2, ..., X_n, ...$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $\sup_n X_n$ and $\limsup_{n\to\infty} X_n$ are also random variables.

[11 marks]

QUESTION 2

(a) Let X be a non-negative random variable such that $\mathbb{E}[X] = 0$. Prove that $X = 0 \quad a.s.$.

[12 marks]

(b) Prove that every non-negative random variable is the limit of an increasing sequence of simple random variables.

[13 marks]

(c) Let A be the set of rational numbers in $[0, 1]$; define the function $f : [0, 1] \to [0, 1]$ by

$$f(x) = I_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Prove that the function f is Lebesgue-integrable but not Riemann-integrable.

[8 marks]

QUESTION 3

(a) Define the conditional expectation of a random variable X given a σ-algebra $\mathcal{G}$, and use this definition to prove the following:

a) if X is $\mathcal{G}$-measurable then $\mathbb{E}[X|\mathcal{G}] = X$

b) if X is independent of $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$

[17 marks]

(b) Define what it means for the σ-algebra $\mathcal{G}$ to be generated by the single random variable Y. In this special case, prove the formula

$$\mathbb{E}[X|\mathcal{G}] = \frac{\int_{-\infty}^{\infty} x f(x, Y) dx}{\int_{-\infty}^{\infty} f(x, Y) dx}$$

where f is the joint probability density of X, Y.

[16 marks]

QUESTION 4

(a) Define the binomial model for a single risky asset.

[6 marks]

(b) Prove the uniqueness of the risk-neutral measure for the binomial model, and derive necessary and sufficient conditions for its existence. Interpret these conditions.

[13 marks]

(c) What can one conclude from the existence and uniqueness of the risk-neutral measure?

[6 marks]

(d) Explain how the risk-neutral measure can be used to obtain the value at time n of a derivative with payoff H at time N ($n \leq N$).

[8 marks]