

DUBLIN CITY UNIVERSITY

SEMESTER ONE REPEAT EXAMINATION 2006-2007

MODULE: MS407/MS407M/SHSAX/SHSAO
Probability and Finance I

COURSE: B. Sc. in Financial and Actuarial Mathematics
B. Sc. in Mathematical Sciences
Study Abroad - Science & Health

YEAR: 3

EXAMINERS: Prof. E. Buffet (ext. 5287)
Prof. T. Hurley
Prof. B. Hanzon

TIME ALLOWED: 2 hours

INSTRUCTIONS: Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions the examiner will mark all questions attempted and then select the highest scoring ones.
Each question carries 33 marks.

REQUIREMENTS: None.

**THE USE OF PROGRAMMABLE OR TEXT STORING
CALCULATORS IS EXPRESSLY FORBIDDEN**

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QUESTION 1

(a) State the axioms of probability theory.

[6 marks]

(b) Use the axioms to prove the following:

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = \lim_{m \rightarrow \infty} \mathbb{P}\left[\bigcup_{n=1}^m A_n\right].$$

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] \leq \sum_{n=1}^{\infty} \mathbb{P}[A_n].$$

[11 marks]

(c) Prove that any countable intersection of almost sure events is almost sure.

[6 marks]

(d) Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $\mu_X = \mathbb{P} \circ X^{-1}$ is a probability measure on $(\mathbb{R}, \mathcal{B})$, where \mathcal{B} is the Borel σ -algebra.

[10 marks]

QUESTION 2

(a) Prove that every positive random variable is the limit of an increasing sequence of simple random variables.

[11 marks]

(b) Outline the steps through which one defines the expectation of a general random variable.

[11 marks]

(c) Let X be a non-negative random variable. Prove that if $\mathbb{E}[X] = 0$, then $X = 0$ is almost surely.

[11 marks]

QUESTION 3

(a) Let B be an event of positive probability in a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) Prove that \mathbb{P}_B defined by

$$\mathbb{P}_B[A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

is a probability measure on (Ω, \mathcal{F}) .

[6 marks]

(c) The expectation \mathbb{E}_B attached to \mathbb{P}_B is given by

$$\mathbb{E}_B[X] = \frac{\mathbb{E}[X I_B]}{\mathbb{P}[B]}$$

where I_B is the indicator of B . Justify briefly this formula.

[8 marks]

(d) Let $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots\}$ be a partition of the sample space Ω into disjoint events of positive probability. For a given random variable X , give the definition of $\mathbb{E}[X|\mathcal{D}]$, the conditional expectation of X given the decomposition \mathcal{D} . Calculate $\mathbb{E}\{\mathbb{E}[X|\mathcal{D}]\}$.

[7 marks]

(e) State and prove the **optimal approximation property** which characterises $\mathbb{E}[X|\mathcal{D}]$ when $\mathbb{E}[X^2] < \infty$.

[12 marks]

QUESTION 4

Consider the general discrete-time model of a financial market on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}\{\omega\} > 0 \quad \forall \omega \in \Omega$. The discounted price of the risky assets at time n is denoted by the vector $\underline{\tilde{S}}_n$, and \mathcal{F}_n is its natural filtration. The discounted value of a self-financing strategy Φ is

$$\tilde{V}_n(\Phi) = V_0(\Phi) + \sum_{m=0}^{n-1} \Phi_m \cdot (\underline{\tilde{S}}_{m+1} - \underline{\tilde{S}}_m).$$

(a) Prove that no arbitrage can exist in the model if there exists a probability \mathbb{P}^* equivalent to \mathbb{P} under which $\underline{\tilde{S}}_n$ is a martingale. (You may assume that $\tilde{V}_n(\phi)$ is also a martingale under \mathbb{P}^*).

[8 marks]

(b) The **Separating Hyperplane Theorem** states that if the subspace V of \mathbb{R}^p and the closed bounded convex set $K \subset \mathbb{R}^p$ do not intersect, then there exists a hyperplane containing V and not intersecting K . In other words, there exists $\underline{\lambda}$ in \mathbb{R}^p such that $\underline{\lambda} \cdot \underline{x} = 0$ for all \underline{x} in V while $\underline{\lambda} \cdot \underline{y} > 0$ for all \underline{y} in K .

(i) Explain how the set of all random variables on the finite sample space Ω can be identified with \mathbb{R}^p . Rephrase the no-arbitrage condition in the form $V \cap K = \phi$ for appropriate V, K .

[8 marks]

(ii) Assuming that there is no arbitrage in the model, use the Separating Hyperplane Theorem to construct a probability \mathbb{P}^* equivalent to \mathbb{P} .

[8 marks]

(c) Check that \tilde{S}_n is a martingale under \mathbb{P}^* .

[9 marks]