

# DUBLIN CITY UNIVERSITY

## SEMESTER ONE REPEAT EXAMINATION 2009-2010

- MODULE: MS407/MS407M/SHSAX/SHSAO  
Probability & Finance 1
- COURSE: M. Sc. in Financial and Actuarial Mathematics  
B. Sc. in Mathematical Sciences  
B. Sc. in Financial and Actuarial Mathematics  
Study Abroad - Science & Health
- YEAR: 1/4
- EXAMINERS: Prof. E. Buffet (ext. 5287)  
Dr. R. Gerrard  
Prof. R. Timoney  
Prof. B. Hanzon
- TIME ALLOWED: 2 hours
- INSTRUCTIONS: Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions the examiner will mark all questions attempted and then select the highest scoring ones.  
Each question carries 33 marks.
- REQUIREMENTS: None

**THE USE OF PROGRAMMABLE OR TEXT STORING  
CALCULATORS IS EXPRESSLY FORBIDDEN**

**DO NOT TURN OVER THIS PAGE UNTIL INSTRUCTED TO DO  
SO.**

## QUESTION 1

(a)

(i) Define the following terms: event, probability measure, random variable,  $\sigma$ -algebra generated by a random variable.

[6 marks]

(ii) Let  $A \subset \Omega$ . Define the indicator  $I_A$ . When is it a random variable? Justify your answer.

[4 marks]

(iii) Assuming that  $I_A$  is a random variable, describe the  $\sigma$ -algebra  $\sigma(I_A)$ .

[4 marks]

(b)

(i) Set-up a probability triple to describe the experiment 'toss a coin three times'.

[4 marks]

(ii) Let  $X$  denote the result of the first toss, and let  $A$  be the event 'more  $H$  than  $T$  occur'. Spell out the  $\sigma$ -algebras  $\sigma(X)$  and  $\sigma(I_A)$ .

[6 marks]

(iii) State, with reasons, which of the following random variables is measurable with respect to  $\sigma(X, I_A)$ , the  $\sigma$ -algebra generated by  $X$  and  $I_A$  jointly:

-  $Y$ : first time  $H$  occurs (with the convention  $Y(TTT) = 4$ )

-  $I_B$  where  $B = \{THH\}$

[9 marks]

## QUESTION 2

(a) Let  $X_n \geq 0$  be a monotone increasing sequence of simple random variables converging to  $X$ , and let  $Y$  be a simple random variable such that  $Y \leq X$ . Finally for  $\varepsilon > 0$  arbitrary but fixed let  $A_n = \{X_n \geq Y - \varepsilon\}$

(i) Prove that  $\lim_{n \rightarrow \infty} \mathbb{P}[A_n] = 1$ .

(ii) Use the indicator of  $A_n$  to prove that  $\lim_{n \rightarrow \infty} \mathbb{E}[X_n] \geq \mathbb{E}[Y]$ .

[18 marks]

(b) Use the above result to prove the following: if  $X_n, Y_n$  are monotone increasing sequences of simple random variables converging to  $X$ , then  $\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \lim_{n \rightarrow \infty} \mathbb{E}[Y_n]$ .

[10 marks]

(c) Comment on the significance of b) in the theory of expectations.

[5 marks]

## QUESTION 3

(a) Define the following:  $\sigma$ -algebra, filtration, martingale.

[6 marks]

(b) State and prove the tower property of conditional expectations.

[8 marks]

(c) Calculate  $\mathbb{E}[X_m | \mathcal{F}_n]$ ,  $m \geq n$ , when  $(X_n)$  is a martingale with respect to  $(\mathcal{F}_n)$ . Prove that martingales have a constant expectation.

[8 marks]

(d) Let  $X_n = \prod_{j=1}^n Y_j$  where the random variables  $Y_j$  are mutually independent. Find a sufficient additional condition which ensures that  $X_n$  is a martingale with respect to a suitable filtration.

[6 marks]

(e) Let  $X$  be a random variable with finite expectation in  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $(\mathcal{F}_n)$  be a filtration of the probability triple. Prove that  $X_n = \mathbb{E}[X | \mathcal{F}_n]$  is a martingale.

[5 marks]

#### QUESTION 4

(a) Describe the binomial model of a financial market with a single risky asset. In this context, define trading strategies and self-financing strategies.

[8 marks]

(b) Give a detailed derivation of the price of a call option with strike  $K$  and maturity  $N = 1$  in the binomial model.

[15 marks]

(c) Explain succinctly how the above result can be extended to general maturities.

[10 marks]