DUBLIN CITY UNIVERSITY

SEMESTER ONE EXAMINATION 2009-2010

MODULE:	MS407/MS407M/SHSAX/SHSAO Probability and Finance 1
COURSE:	M. Sc. in Financial and Industrial Mathematics B. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. R. Timoney Prof. B. Hanzon Dr. R. Gerrard
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions the examiner will mark all questions attempted and then select the highest scoring ones. Each question carries 33 marks.
REQUIREMENTS:	None.

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) Define the following: Borel σ-algebra $\mathcal{B}$, Borel function $f : \mathbb{R} \to \mathbb{R}$. Prove that $f(x) = x^2$ is a Borel function.

[7 marks]

(b) Prove that if X, Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, so is $X + Y$.

[6 marks]

(c) Prove that if X is a random variable, then so is $f(X)$ whenever f is a Borel function.

[6 marks]

(d) Use the above to prove that if X, Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, so is XY.

[6 marks]

(e) What does it mean for a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ to be measurable with respect to the σ-algebra $\mathcal{G} \subset \mathcal{F}$? Prove that if X is measurable with respect to the trivial σ-algebra $\mathcal{G} = \{\phi, \Omega\}$, then X is constant.

[8 marks]

QUESTION 2

(a) Prove that every non-negative random variable can be obtained as the limit of an increasing sequence of simple random variables.

[13 marks]

(b) Consider the case where $\Omega = [0, 1]$ and $\mathcal{F} = \mathcal{B}$ so that random variables are Borel functions. Illustrate graphically the above approximation scheme and contrast it with the one used in the standard (Riemann) theory of integration.

[8 marks]

(c) Consider $X = I_{[0,1] \bigcap \mathbb{Q}}$. Is this a random variable on the previous space $(\Omega, \mathcal{F})$? Explain why, and calculate its expectation if $\mathbb{P}$ is the Lebesgue measure. Is X integrable in the sense of Riemann? Explain why.

[12 marks]

QUESTION 3

Let B be an event of positive probability in a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$.
(a) Prove that $\mathbb{P}_B$ defined by $\mathbb{P}_B[A] = \frac{\mathbb{P}[A \bigcap B]}{\mathbb{P}[B]}$ is a probability measure on $(\Omega, \mathcal{F})$.

[6 marks]

(b) The expectation $\mathbb{E}_B$ attached to $\mathbb{P}_B$ is given by $\mathbb{E}_B[X] = \frac{\mathbb{E}[XI_B]}{\mathbb{P}[B]}$ where I_B is the indicator of B. Justify briefly this formula.

[8 marks]

(c) Let $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, ...\mathcal{D}_n, ...\}$ be a partition of the sample space Ω into disjoint events of positive probability. For a given random variable X, give the definition of $\mathbb{E}[X|\mathcal{D}]$, the conditional expectation of X given the decomposition $\mathcal{D}$. Calculate $\mathbb{E}\{\mathbb{E}[X|\mathcal{D}]\}$.

[7 marks]

(d) State and prove the **optimal approximation property** which characterises $\mathbb{E}[X|\mathcal{D}]$ when $\mathbb{E}[X^2] < \infty$.

[12 marks]

QUESTION 4

(a) Define the following terms: american option, intrinsic value of an american option, exercise region, continuation region.

[6 marks]

(b) Justify the recursive formula

$$V_n^a = \max\left(H_n, \frac{1}{1+r}\mathbb{E}^*[V_{n+1}^a|\mathcal{F}_n]\right)$$

for the value at time n of an american option with intrinsic value H_n.

[8 marks]

(c) Write down the form taken by the above in the case of a put option with strike K on a stock following a binomial model with rates of change u and d, so that the risk-neutral probability that the stock goes up is

$$p^* = \frac{1+r-d}{u-d}.$$

[4 marks]

(d) A stock is worth 16EUR at time zero. Calculate the value at time zero of an american put with strike 14EUR and maturity 3, assuming that the stock price follows a binomial model with $u = \frac{3}{2}, d = \frac{1}{2}, r = \frac{1}{4}$. Determine also the exercise boundary.

[15 marks]