DUBLIN CITY UNIVERSITY

SEMESTER ONE REPEAT EXAMINATION 2007-2008

MODULE:	MS407/MS407M/SHSAX/SHSAO Probability & Finance 1
COURSE:	M. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences B. Sc. in Financial and Actuarial Mathematics Study Abroad - Science & Health
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. Hurley Prof. B. Hanzon
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) State the axioms of Probability Theory.

[6 marks]

(b) Let Ω be an **uncountable** sample space; define $\mathcal{C}$ to be the class of all the subsets of Ω that are either **finite** or **cofinite** (that is to say their complement is finite). Finally, specify a function $K : \mathcal{C} \to \mathbb{R}$ by

$$K[C] = \begin{cases} 0 & \text{if C is finite,} \\ \\ 1 & \text{if C is cofinite} \end{cases}$$

Answer the following questions, providing full justifications for your answers:

(i) Is $\mathcal{C}$ an algebra?

[8 marks]

(ii) Is $\mathcal{C}$ a σ-algebra?

[4 marks]

(iii) Is K well defined?

[4 marks]

(iv) Is K finitely additive?

[7 marks]

(v) Is K σ-additive on $\mathcal{C}$?

[4 marks]

QUESTION 2

(a) State the Monotone Convergence Theorem.

[4 marks]

(b) State and prove Fatou's lemma.

[13 marks]

(c) State and prove the Dominated Convergence Theorem.

[16 marks]

QUESTION 3

(a) Describe in detail, but without proof, the construction of the conditional expectation $\mathbb{E}[X|\mathcal{G}]$, where $\mathcal{G}$ is a σ-algebra contained in the set of events $\mathcal{F}$.

[15 marks]

(b) Using the above definition, prove the following properties:

(i) $\mathbb{E}\{\mathbb{E}[X|\mathcal{G}]\} = \mathbb{E}[X]$;

(ii) $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$ whenever Y is $\mathcal{G}$-measurable and bounded;

(iii) $\mathbb{E}\{\mathbb{E}[X|\mathcal{G}]|\mathcal{H}\} = \mathbb{E}[X|\mathcal{H}] \quad$ if $\mathcal{H} \subset \mathcal{G}$.

[18 marks]

QUESTION 4

Consider the general discrete-time model of a financial market on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}\{\omega\} > 0 \quad \forall\, \omega \in \Omega$. The discounted price of the risky assets at time n is denoted by the vector $\underline{\tilde{S}_n}$, and $\mathcal{F}_n$ is its natural filtration. The discounted value of a self-financing strategy Φ is

$$\tilde{V}_n(\Phi) = V_0(\Phi) + \sum_{m=0}^{n-1} \underline{\Phi}_m \cdot (\underline{\tilde{S}}_{m+1} - \underline{\tilde{S}}_m).$$

(a) Prove that no arbitrage can exist in the model if there exists a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which $\underline{\tilde{S}_n}$ is a martingale. (You may assume that $\tilde{V}_n(\Phi)$ is also a martingale under $\mathbb{P}^*$).

[8 marks]

(b) The **Separating Hyperplane Theorem** states that if the subspace V of $\mathbb{R}^p$ and the closed bounded convex set $K \subset \mathbb{R}^p$ do not intersect, then there exists a hyperplane containing V and not intersecting K. In other words, there exists $\underline{\lambda}$ in $\mathbb{R}^p$ such that $\underline{\lambda} \cdot \underline{x} = 0$ for all $\underline{x}$ in V while $\underline{\lambda} \cdot y > 0$ for all $\underline{y}$ in K.

(i) Explain how the set of all random variables on the finite sample space Ω can be identified with $\mathbb{R}^p$. Rephrase the no-arbitrage condition in the form $V \bigcap K = \phi$ for appropriate V, K.

[8 marks]

(ii) Assuming that there is no arbitrage in the model, use the Separating Hyperplane Theorem to construct a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$.

[8 marks]

(iii) Check that $\underline{\tilde{S}}_n$ is a martingale under $\mathbb{P}^*$.

[9 marks]