

DUBLIN CITY UNIVERSITY

SEMESTER ONE EXAMINATION 2007/2008

MODULE: MS407/MS407M/SHSAX/SHSAO
Probability & Finance 1

COURSE: M. Sc. in Financial and Actuarial Mathematics
B. Sc. in Mathematical Sciences
B. Sc. in Financial and Actuarial Mathematics
Study Abroad - Science & Health

YEAR: 1/4

EXAMINERS: Prof. E. Buffet (ext. 5287)
Prof. B. Hanzon
Prof. T. Hurley

TIME ALLOWED: 2 hours

INSTRUCTIONS: Attempt any THREE questions.
All questions carry equal marks.

Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones

REQUIREMENTS: None

**THE USE OF PROGRAMMABLE OR TEXT STORING
CALCULATORS IS EXPRESSLY FORBIDDEN**

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SO.**

QUESTION 1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple.

(a) Define I_A the indicator of A , where A is a subset of Ω . When is I_A a random variable? Justify your answer.

[5 marks]

(b) Let X, Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $X + Y$ is also a random variable.

[7 marks]

(c) Prove that $\mu_X = \mathbb{P}_0 X^{-1}$ is a probability measure on $(\mathbb{R}, \mathcal{B})$, where \mathcal{B} is the Borel σ -algebra.

[7 marks]

(d) Define Borel functions and prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function and X is a random variable, then $f(X)$ is a random variable.

[7 marks]

(e) Prove that $f(x) = x^2$ is a Borel function, and thus prove that if X and Y are random variables, so is XY .

[7 marks]

QUESTION 2

(a) Explain in detail (but without proof) the steps through which the definition of $\mathbb{E}[X]$ is arrived at for a general random variable X .

[13 marks]

(b) State and prove the Monotone Convergence Theorem.

[20 marks]

QUESTION 3

(a) Show by means of a two-dimensional example that, in general, orthogonal projections do not preserve positivity.

[4 marks]

(b) Let $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}\}$, \mathbb{P} specified through $p_{\omega_1}, p_{\omega_2}$.

(i) Explain how $L^2(\mathcal{F})$ can be identified with \mathbb{R}^2 .

(ii) Find a smaller σ -algebra $\mathcal{G} \subset \mathcal{F}$ and explain which subspace of \mathbb{R}^2 corresponds to $L^2(\mathcal{G})$.

(iii) Calculate the orthogonal projection of a general random variable X onto that subspace, and thus check that the projection preserves positivity.

[12 marks]

(c) Prove that, in the case of a general probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and a general σ -algebra $\mathcal{G} \subset \mathcal{F}$, $\mathbb{E}[X|\mathcal{G}] \geq 0$ a.s. if $X \geq 0$ a.s.. You may find it useful to consider the events

$$G_n = \left\{ \omega : \mathbb{E}[X|\mathcal{G}](\omega) \leq -\frac{1}{n} \right\}.$$

[17 marks]

QUESTION 4

(a) Set up the general discrete-time model of a financial market on a finite probability space, and in this context define the following terms: trading strategy, self-financing strategy, arbitrage strategy.

[10 marks]

(b) State (without proof) the separating hyperplane theorem.

[5 marks]

(c) Consider a binomial model with interest rate $\frac{1}{4}$ per period; assume that the price S_{n+1} of the risky asset at time $n+1$ is either $\frac{3}{2}S_n$ or $\frac{1}{2}S_n$. Compute the value at time zero of an American put with strike price €9 and maturity $N = 3$ on a stock with initial value €8. Identify the exercise boundary.

[18 marks]