DUBLIN CITY UNIVERSITY

SEMESTER ONE REPEAT EXAMINATION 2003 – 2004

MODULE:	MS407 Probability & Finance 1
COURSE:	M. Sc. in Financial & Industrial Mathematics B. Sc. in Mathematical Sciences B. Sc. in Financial & Actuarial Mathematics
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. Hurley Prof. M. H. A. Davis
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks.
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) Prove that any countable union of countable sets is countable.

[7 marks]

(b) Prove that any countable intersection of almost sure events is almost sure.

[7 marks]

(c) Let A be a subset of the sample space Ω. Define the indicator of A. Is it a random variable? Prove your assertion.

[7 marks]

(d) Define μ_X, the probability distribution of the random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that μ_X is a probability measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the collection of the Borel subsets of $\mathbb{R}$.

[12 marks]

QUESTION 2

(a) Explain in detail, but without proof, the steps through which one defines the expectation of a random variable on a general probability triple.

[17 marks]

(b) Discuss briefly the connection between the above construction and that of the Lebesgue integral. Prove that the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

is Lebesgue-integrable but not Riemann-integrable.

[16 marks]

QUESTION 3

(a) Define the following terms: σ-algebra, filtration, martingale.

[6 marks]

(b) State and prove the tower property of conditional expectations.

[8 marks]

(c) Calculate $\mathbb{E}[X_m|\mathcal{F}_n], m \geq n$, when (X_n) is a martingale with respect to $(\mathcal{F}_n)$. Prove that martingales have a constant expectation.

[8 marks]

(d) Let

$$X_n = \prod_{j=1}^{n} Y_j$$

where the random variables Y_j are mutually independent. Find a sufficient additional condition which ensures that X_n is a martingale with respect to a suitable filtration.

[6 marks]

(e) Let X be a random variable with finite expectation in $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_n)$ be a filtration of the probability triple. Prove that $X_n = \mathbb{E}[X|\mathcal{F}_n]$ is a martingale.

[5 marks]

QUESTION 4

(a) Describe the binomial model of a financial market with a single risky asset. In this context, define trading strategies and self-financing strategies.

[8 marks]

(b) Give a detailed derivation of the price of a call option with strike K and maturity $N = 1$ in the binomial model.

[15 marks]

(c) Explain succinctly how the above result can be extended to general maturities.

[10 marks]