DUBLIN CITY UNIVERSITY

SEMESTER ONE EXAMINATION 2003 – 2004

MODULE:	MS407 Probability & Finance 1
COURSE:	M. Sc. in Financial & Industrial Mathematics B. Sc in Mathematical Sciences B. Sc. in Financial & Actuarial Mathematics
YEAR:	1/4
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. Hurley Prof. M. H. A. Davis
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any THREE questions. All questions carry equal marks.
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

(a) State the axioms of Probability Theory

[6 marks]

(b) Prove the monotone continuity property

$$\mathbb{P}\left[\bigcup_{j=1}^{\infty} A_j\right] = \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right]$$

[8 marks]

(c) Consider the experiment "toss a fair coin twice":

(i) Set up a probability triple to describe this experiment

[4 marks]

(ii) Let the random variable X denote the total number of "Heads". Construct explicitly the σ-algebra and $\sigma(X)$.

[4 marks]

(iii) Give an example of a random variable Y, which is measurable with respect to $\sigma(X)$ but such that $\sigma(X) \neq \sigma(Y)$.

[5 marks]

(iv) With X as in (ii), let $Z = I_{\{X>1\}}$. Obtain a formula for $\mathbb{E}[X|Z]$.

[6 marks]

QUESTION 2

(a) State the monotone convergence theorem.

[4 marks]

(b) State and prove Fatou's lemma.

[13 marks]

(c) State and prove the dominated convergence theorem.

[16 marks]

QUESTION 3

(a) Define the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ where X is a random variable on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$.

[6 marks]

(b) Prove that if the random variable X is almost surely positive, so is $\mathbb{E}[X|\mathcal{G}]$. You may find it useful to consider the events

$$A_n = \{\omega : \mathbb{E}[X|\mathcal{G}](\omega) \leq -\frac{1}{n}\}$$

and their indicators I_{A_n}.

[10 marks]

(c) When is a random variable X said to be independent of a σ-algebra $\mathcal{G}$? What is the value of $E\ [X|\mathcal{G}]$ in such a case? Prove your assertion.

[6 marks]

(d) Define the conditional expectation $\mathbb{E}[X|Y]$ and prove that if X and Y have the joint probability density $f(x, y)$, then

$$\mathbb{E}[X|Y] = \frac{\int_{-\infty}^{\infty} x f(x,y)\, dx}{\int_{-\infty}^{\infty} f(x,y)\, dx}.$$

[11 marks]

QUESTION 4

(a) Set up the general model of discrete-time financial market with finite time horizon. Define the following terms: self-financing strategy, arbitrage strategy, complete market.

[10 marks]

(b) Give precise statements of two theorems relating arbitrage, risk-neutral probability measures and completeness. These two theorems amount to four implications. Prove any one of these implications.

[11 marks]

(c) Define the binomial model of a Financial Market with a single risky asset. Construct a risk-neutral probability $\mathbb{P}^*$ for this model, and check that is is unique

[12 marks]