DUBLIN CITY UNIVERSITY

SEMESTER ONE REPEAT EXAMINATION 2007-2008

MODULE:	MS308/SHSAX/SHSAO Stochastic Modelling
COURSE:	B. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences Study Abroad - Science & Health
YEAR:	3
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. C. Hurley Dr. R. Gray Dr. P. King
TIME ALLOWED:	3 hours
INSTRUCTIONS:	Candidates who are registered for Actuarial Exemptions must answer **all** four questions. Candidates who are **not** registered for exemptions should attempt any **three** out of four questions. Each question carries 25 marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	Candidates should provide their own electronic calculators. Mathematics tables will be provided by the university; Actuarial Tables are **not** required.

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

A rating agency grades financial institutions for credit-worthiness; it uses four grades listed in decreasing order of merit: A, B, C and D (default). The grades are revised yearly; all downgrades occur with probability α, upgrades from B or C occur with probability β. If a firm is in default it remains in that state unless it is rescued by an outside investor, in which case it can move to the credit class B or C with equal probability ε.

(a) Set up a discrete-time Markov chain to describe the above; draw the transition graph and write down the transition matrix of the chain. Is the chain irreducible, is it aperiodic? Justify your answers.

[7 marks]

(b) Calculate the stationary probability distribution of the chain (up to normalisation) in terms of $\alpha, \beta, \varepsilon$.

[9 marks]

(c) Data shows that, in the long term, the number of firms in default is one fifth of the number of firms with a C rating and one sixth of the number of firms with a B rating. Use this to obtain the numerical value of the stationary probability distribution.

[9 marks]

QUESTION 2

Consider the most general two-state discrete-time Markov chain.

(a) Write down its transition matrix $\mathbf{P}$ and calculate its eigenvalues and right-eigenvectors.

[7 marks]

(b) Hence calculate explicitly $\mathbf{P}^n$ for all integers n.

[7 marks]

(c) Calculate $\lim_{n\to\infty} \mathbf{P}^n$, for <u>all</u> values of the transition probabilities.

[6 marks]

(d) Calculate the stationary probability distribution of the chain and comment on the result.

[5 marks]

QUESTION 3

Consider a population that grows only through immigration; accordingly the transition rate from state j to state $j+1$ is λ for all $j \epsilon \{0, 1, 2, ...\}$. On the other hand the population faces extinction from natural disasters, reflected in a transition rate from state j to state 0 equal to δ for all $j \epsilon \{1, 2, 3, ...\}$.

(a) Draw the transition graph and write down the generator of a Markov jump process modelling the above situation in continuous time. Is the process conservative? Explain why.

[6 marks]

(b) Compute the stationary probability distribution of the process.

[9 marks]

(c) Write down the forward equations for this process and compute the extinction probabilites $p_{j0}(t), \quad j > 0$.

[10 marks]

QUESTION 4

(a) Consider a Markov jump process with transition matrix $\mathbf{P}(t)$ and generator G. Write down the backward equation and check that any solution of the integral equation

$$p_{ij}(t) = \delta_{ij} e^{g_{ii} t} + \int_0^t \sum_{\substack{k \\ k \neq i}} e^{s g_{ii}} g_{ik} p_{kj}(t-s) ds$$

is also a solution of the backward equation.

[10 marks]

(b) Let $T = \min\{t \geq 0 : X_t \epsilon A\}$ be the first hitting time of set A by the Markov jump process X_t, and let $m_i = \mathbb{E}[T | X_0 = i]$ be the mean hitting time of A starting from state i. Prove that the numbers m_i obey the equations

$$m_i = 0 \quad i \in A$$

$$\sum_j g_{ij} m_j = -1 \quad i \notin A.$$

[15 marks]