DUBLIN CITY UNIVERSITY

SEMESTER ONE EXAMINATION 2007-2008

MODULE:	MS308/SHSAX/SHSAO Stochastic Modelling
COURSE:	B. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences Study Abroad - Science & Health
YEAR:	3
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. C. Hurley Dr. R. Gray Dr. P. King
TIME ALLOWED:	3 hours
INSTRUCTIONS:	Candidates who are registered for Actuarial Exemptions must answer **all** four questions. Candidates who are **not** registered for exemptions should attempt any **three** out of four questions. Each question carries 25 marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	Candidates should provide their own electronic calculators. Mathematics tables will be provided by the university; Actuarial Tables are **not** required.

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

A motor insurance company classifies its policy holders as poor, average or good drivers. Any accident incurred during a year results in a downgrade the following year, whereas it takes two successive accident-free years to merit an upgrade. Suppose that the probability of suffering an accident during a period of one year is $\frac{3}{10}$, $\frac{2}{10}$ or $\frac{1}{10}$ depending on the driver's merit class.

(a) Construct a discrete-time Markov chain to describe the status of a representative driver under this scheme (a five-state model is suggested). Is the chain irreducible, is it aperiodic? Justify your answers.

[9 marks]

(b) Compute the stationary probability distribution of the chain.

[8 marks]

(c) The average yearly claim per policy is €1,200, €800 or €150, depending on the merit class of the driver, whereas the corresponding yearly premiums are C, 0.7C or 0.5C. What is the smallest value of C for which the scheme is viable?

[8 marks]

QUESTION 2

(a) Define the following terms for a Markov chain: stationary distribution, limiting distribution, recurrent state.

[6 marks]

(b) Let $p_{ij}(n)$ denote the n-step transition probabilities of a finite Markov chain. Let

$$m_j(n) = \min_i p_{ij}(n), \quad M_j(n) = \max_i p_{ij}(n).$$

Prove that for each j the sequence $m_j(n)$ is increasing in n whereas $M_j(n)$ is decreasing in n.

[4 marks]

(c) Use the fact that for a finite irreducible aperiodic Markov chain there exists a real number ε and an integer n_0 such that

$$p_{ij}(n) \geq \varepsilon > 0 \quad \forall i\, j, \quad \forall n \geq n_0$$

to prove the following inequality:

$$m_j(n_0 + l) \geq (1 - \varepsilon) m_j(l) + \varepsilon p_{jj}(2l) \quad \forall l > 0.$$

[7 marks]

(d) Hence prove that $p_{ij}(n)$ converges as $n \to \infty$ to a limit ν_j that is independent of i and that moreover the rate of convergence is exponential.

[8 marks]

QUESTION 3

A rating agency grades financial institutions for credit-worthiness; it uses four grades listed in order of decreasing merit: A, B, C and D (default). Grades are revised continuously, changes always being to the next grade (up or down), except for firms in default which can (if rescued by an outside investor) jump either to grade B or to grade C with equal probability.

(a) Set up a continuous-time Markov jump process to describe the above scheme. Write down the most general generator that fits the scheme (it should have six parameters).

[4 marks]

(b) Suppose that for firms with a B-rating downgrades are twice as likely as upgrades, whereas upgrades and downgrades are equally likely for firms with a C-rating. Use this to reduce the number of parameters in the generator to four.

[3 marks]

(c) Suppose moreover that, on average, firms holding an A-rating retain it for four consecutive years, whereas firms in default have to wait six months before being rescued. Use this to reduce the number of parameters in the generator to two.

[4 marks]

(d) Suppose finally that the mean time to default is nine years for A-rated firms whereas it is thirty-three months for C-rated firms. Obtain the explicit form of the generator.

[8 marks]

(e) Calculate the stationary probability distribution of the process.

[6 marks]

QUESTION 4

(a) Define the following terms: Markov jump process, jump time, holding time.

[6 marks]

(b) Let X_t, $t \geq 0$ be a Markov jump process with generator G. Compute

$$\mathbb{P}\left[X_{\frac{ks}{2^n}} = i,\ k = 1, 2, ..., 2^n | X_0 = i\right]$$

and thus obtain the probability distribution of H_0, the first holding time of the process.

[7 marks]

(c) Compute

$$\mathbb{P}[X_{s+h} = j, s < H_0 \leq s + h | X_0 = i], \quad i \neq j$$

and thus obtain the joint distribution of X_{H_0} and H_0.

[8 marks]

(d) What can you conclude from ii) and iii) taken together?

[4 marks]