DUBLIN CITY UNIVERSITY

SEMESTER ONE REPEAT EXAMINATION 2006-2007

MODULE:	MS308/SHSAX/SHSAO Stochastic Modelling
COURSE:	B. Sc. in Financial and Actuarial Mathematics B. Sc. in Mathematical Sciences Study Abroad - Science & Health
YEAR:	3
EXAMINERS:	Prof. E. Buffet (ext. 5287) Prof. T. C. Hurley Dr. R. Gray Dr. P. King
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Candidates who are registered for Actuarial Exemptions must answer **all** four questions. Candidates who are **not** registered for exemptions should attempt any **three** out of four questions. Each question carries 25 marks. Please note that where a candidate answers more than the required number of questions, the examiner will mark all questions attempted and then select the highest scoring ones
REQUIREMENTS:	Candidates should provide their own electronic calculators. Mathematics tables will be provided by the university; Actuarial Tables are **not** required.

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

DO NOT TURN OVER THIS PAGE UNTIL INSTRUCTED TO DO SO.

QUESTION 1

Let $Y_1, Y_2, ..., Y_n, ...$ be independent identically distributed random variables with the probability distribution

$$\mathbb{P}[Y_n = 1] = \alpha, \quad \mathbb{P}[Y_n = -1] = 1 - \alpha, \quad 0 < \alpha < 1.$$

A moving average is defined as

$$X_n = \frac{1}{3}\left(Y_n + Y_{n-1} + Y_{n-2}\right), \quad n \geq 3.$$

(a) Prove that X_n is a stationary process and obtain its probability distribution.

[4 marks]

(b) Prove that X_{n+3} is independent of X_n, and thus compute the 3-step transition matrix $P(3)$ with entries

$$p_{ij}(3) = \mathbb{P}(X_{n+3} = j | X_n = i).$$

[7 marks]

(c) Compute the 1-step transition matrix of X_n.

[8 marks]

(d) Write down the special form that the matrices $P(1)$ and $P(3)$ take when $\alpha = \frac{1}{2}$. Is the moving average a Markov chain? Explain why.

[6 marks]

QUESTION 2

Let $X_n, n = 0, 1, 2, ...$ be a discrete-time Markov chain.

(a) Define the following: recurrent state, mean recurrence time. Let

$$\alpha_j(n) = \mathbb{P}(X_1 \neq j, X_2 \neq j, ..., X_n \neq j | X_0 = j).$$

Prove that

$$1 - p_{jj}(n) = \alpha_j(n) + \sum_{l=1}^{n-1} \alpha_j(n-l) p_{jj}(l).$$

[6 marks]

(b) Prove that, if j is recurrent

$$\alpha_j(n) = \sum_{k=n+1}^{\infty} f_{jj}(k).$$

[3 marks]

(c) Extend the definition of $\alpha_j(n)$ to $n = 0$ in a manner consistent with b) and thus rewrite the equation obtained in a) in the simpler form

$$\sum_{m=0}^{n} \alpha_j(m) p_{jj}(n-m) = 1.$$

[3 marks]

(d) Compute $\sum_{m=0}^{\infty} \alpha_j(m)$

[5 marks]

(e) Hence prove the formula $\mu_j \nu_j = 1$ where $\nu_j = \lim_{n\to\infty} p_{jj}(n)$ and μ_j is the mean recurrence time of state j.

[8 marks]

QUESTION 3

(a) Write down the transition graph and the generator of a pure birth process with state space $\{0, 1, 2, ..., n, ...\}$ and transition rates $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n, ...$

[4 marks]

(b) Define conservative and explosive Markov jump processes. Prove that if $\sum_{j=0}^{\infty} \frac{1}{\alpha_j} < \infty$, the above pure birth process is explosive.

[7 marks]

(c) Prove that, if $\varepsilon_j \geq 0$

$$\prod_{j=0}^{\infty} (1 + \varepsilon_j) < \infty \Leftrightarrow \sum_{j=0}^{\infty} \varepsilon_j < \infty$$

[5 marks]

(d) Using the above, prove that a pure birth process with transition rates α_j that obey $\sum_{j=0}^{\infty} \frac{1}{\alpha_j} = \infty$ is conservative.

[9 marks]

QUESTION 4

A disability benefit scheme is modelled in continuous time by a Markov jump process with states A (active), T (temporarily disabled), P (permanently disabled), and D (dead). The transition rates are as follows:

$A \to T : 3\lambda \quad T \to A : 5\lambda \quad P \to D : 2\alpha$
$A \to P : \lambda \quad T \to P : 2\lambda$
$A \to D : \alpha \quad T \to D : \alpha$

where α, λ are two positive numbers.

(a) Draw the transition graph and write down the generator of the process.

[4 marks]

(b) What is the probability that a member who joins the scheme in the Active state will never draw any disability benefit? What is the probability that the same member will draw no benefit over $(0, t)$?

[6 marks]

(c) The average duration of a disability benefit is half a year (if a member is temporarily disabled on several occasions, these are regarded as separate benefits). What does this imply for the parameters α, λ? Use this to write the generator in terms of λ only.

[5 marks]

(d) The average time until death for a member who joins in the Active state is 46 years. Use this (and c) to obtain a cubic equation for λ. Prove that the equation has a unique solution in the appropriate range.

[10 marks]