

# DUBLIN CITY UNIVERSITY

## SEMESTER ONE EXAMINATION 2004

MODULE: MS401  
Numerical Linear Algebra and Applics.

COURSE: B. Sc. in Mathematical Sciences,  
B. Sc. in Financial and Actuarial Mathematics,  
M. Sc. in Financial and Industrial Mathematics.

YEAR: 4

EXAMINERS: Mr. P. Cooper,  
Prof. M. H. A. Davies,  
Prof. T. Hurley,  
Dr. E. O'Riordan (ext. 5386).

TIME ALLOWED: 2 hours

INSTRUCTIONS: Attempt any three questions. All questions carry equal marks.

REQUIREMENTS: None

**THE USE OF PROGRAMMABLE OR TEXT STORING  
CALCULATORS IS EXPRESSLY FORBIDDEN**

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### QUESTION 1

Consider the following class of elliptic problems

$$\begin{aligned} Lu &\equiv -(u_{xx} + u_{yy}) + a_1 u_x + a_2 u_y + bu = f, & (x, y) \in \Omega = (0, 1) \times (0, 1); \\ u &= g, & (x, y) \in \partial\Omega \\ a_1(x, y) &\geq \alpha_1 > 0, \quad a_2(x, y) \geq \alpha_2 > 0, & b(x, y) \geq \beta > 0, \quad (x, y) \in \bar{\Omega}. \end{aligned}$$

Discretize this problem using the upwind finite difference method

$$\begin{aligned} L^{N,M} z &\equiv -(\delta_x^2 z + \delta_y^2 z) + a_1 D_x^- z + a_2 D_y^- z + bz = f, & (x_i, y_j) \in \Omega^{N,M} \\ z &= g, & (x_i, y_j) \in \partial\Omega^{N,M} \end{aligned}$$

where  $\bar{\Omega}^{N,M} = \{(x_i, y_j)\}_{i,j=0}^{N,M} \subset \bar{\Omega}$  is an arbitrary non-uniform mesh.

(a) State and prove a comparison principle for the differential operator  $L$ . Deduce the following upper bound on the solution  $u$

$$\|u\| \leq \frac{\|f\|}{\beta} + \|g\|.$$

[8 marks]

(b) Show that the finite difference operator  $L^{N,M}$  satisfies a discrete comparison principle. Deduce that the discrete solution  $z$  is bounded by the data  $f$  and  $g$ .

[4 marks]

(c) Derive an expression for the truncation error  $|L^{N,M}(u - z)|$  on a uniform mesh. Prove that, on a uniform mesh,

$$\|u - z\|_{\bar{\Omega}^{N,M}} \leq C_1 N^{-1} + C_2 M^{-1}.$$

State explicitly how the constants  $C_1$  and  $C_2$  depend on the maximum value of certain partial derivatives of the solution.

[10 marks]

(d) Consider the case of  $a_2(x, y) = b(x, y) \equiv 0$ . Establish a comparison principle for the differential operator in this case and deduce the upper bound

$$\|u\| \leq \frac{\|f\|}{\alpha_1} + \|g\|.$$

Hint: Associated with any  $v \in C^2$ , consider an associated function  $w$  with  $v(x, y) = e^{\theta x} w(x, y)$  and choose  $\theta$  appropriately.

[11 marks]

## QUESTION 2

(a) Using Fourier analysis, examine the stability of the Lax–Wendroff scheme

$$D_t^+ U(x_j, t_n) = cD_x^0 U(x_j, t_n) + \frac{c^2 k}{2} \delta_x^2 U(x_j, t_n)$$

on a uniform rectangular mesh for the partial differential equation

$$u_t = cu_x.$$

Show that the local truncation error is second order in both space and time.

[16 marks]

(b) Consider the Crank–Nicolson scheme

$$L_\theta^N U(x_j, t_n) \equiv D_t^- U(x_j, t_n) - \frac{1}{2} \delta_x^2 U(x_j, t_{n-1}) - \frac{1}{2} \delta_x^2 U(x_j, t_n) = 0, \text{ in } G^{h,k}$$
$$U = u, \quad \text{in } \partial G^{h,k}$$

on a uniform mesh  $\bar{G}^{h,k} = \{(x_j, t_n) | x_j = jh, t_n = nk\}_{j,n=0}^{N,M}$  for the heat equation

$$u_t - u_{xx} = 0, \text{ in } G = (0, 1) \times (0, 1]$$
$$u(x, 0) = g(x), \quad x \in [0, 1]; \quad u(0, t) = u(1, t) = 0, \quad t > 0;$$
$$\text{and } g(0) = g(1) = g'(0) = g'(1) = 0.$$

Write this scheme in the matrix form

$$A\mathbf{u}_j = B\mathbf{u}_{j-1}$$

where  $\mathbf{u}_j = (U(x_1, t_j), \dots, U(x_{N-1}, t_j))$ . Write out the elements of the matrices  $A$  and  $B$ . Using Taylor expansions about the point  $(x_i, t_{j-1/2})$  determine the order of the local truncation error. Using Fourier analysis, examine the stability of this scheme.

[17 marks]

### QUESTION 3

(a) Consider the system

$$S\mathbf{x} = T\mathbf{x} + \mathbf{b}$$

and suppose that  $S$  is nonsingular and that  $\|S^{-1}T\| < 1$ . Show that

$$\|\mathbf{x}\| \leq \frac{\|S^{-1}\mathbf{b}\|}{1 - \|S^{-1}T\|}$$

[6 marks]

(b) State and prove Gershgorin's circle theorem. Let  $A = (a_{ij})$  be a  $n \times n$  real matrix. Assume that  $A$  is strictly row diagonally dominant. Show that  $A$  is nonsingular and that

$$\rho(A^{-1}) \leq \frac{1}{\min_i (|a_{ii}| - \sum_{k=1, k \neq i}^n |a_{ik}|)}.$$

[10 marks]

(c) Given an  $n \times n$  matrix  $A$ , with the properties that the diagonal terms are nonnegative and that the off-diagonal terms are nonpositive, show that  $A$  can be written in the form  $A = sI - B$ , where  $s > 0$ ,  $B \geq 0$ . If  $s$  can be chosen so that  $s > \rho(B)$ , then show that  $A$  is nonsingular and  $A^{-1} \geq 0$ .

[10 marks]

(d) Deduce an expression for the optimal choice for the SOR parameter  $\omega$  from Young's equation

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2,$$

where  $\lambda$  are the eigenvalues of the SOR iteration matrix  $G_{SOR}$  and  $\mu$  are the eigenvalues of the Jacobi iteration matrix  $G_J$ .

[7 marks]

#### QUESTION 4

(a) Show that the eigenvalues of a real symmetric matrix  $A$  are all real. Show that a real symmetric matrix is positive definite if and only if all the eigenvalues are positive. Show also if  $A$  is positive definite then  $A^k$  is positive definite for any integer  $k$ .

[12 marks]

(b) Assume in all the remaining parts of this question that  $A$  is a real symmetric positive definite matrix and  $\mathbf{b}$  is an arbitrary vector. The Conjugate Gradient method for solving the equation  $A\mathbf{x} = \mathbf{b}$  may be written in the following form:

Let  $\mathbf{x}_0$  be given, and let  $\mathbf{d}_0 = -\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$ . For  $k = 0, 1, 2, \dots$ , compute

$$\alpha_k = -\frac{\mathbf{r}_k \cdot \mathbf{d}_k}{\mathbf{d}_k \cdot A\mathbf{d}_k}, \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \quad \text{and} \quad \mathbf{r}_{k+1} = A\mathbf{x}_{k+1} - \mathbf{b};$$

stop if  $\mathbf{r}_{k+1}$  is sufficiently small; otherwise define

$$\beta_k = \frac{\mathbf{r}_{k+1} \cdot A\mathbf{d}_k}{\mathbf{d}_k \cdot A\mathbf{d}_k}, \quad \mathbf{d}_{k+1} = -\mathbf{r}_{k+1} + \beta_k \mathbf{d}_k$$

and continue.

For the Conjugate Gradient method, it is known that the subspace spanned by the search directions is identical to the subspace spanned by the residuals, which in turn is identical to the Krylov subspace  $[\mathbf{r}_0, A\mathbf{r}_0, \dots, A^k \mathbf{r}_0]$ .

1. Show that

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \alpha_k A\mathbf{d}_k$$

and that

$$\alpha_k = \frac{\mathbf{r}_k \cdot \mathbf{r}_k}{\mathbf{d}_k \cdot A\mathbf{d}_k}, \quad \beta_k = \frac{\mathbf{r}_{k+1} \cdot \mathbf{r}_{k+1}}{\mathbf{r}_k \cdot \mathbf{r}_k}.$$

[8 marks]

2. Show that  $\mathbf{r}_i \cdot \mathbf{r}_j = 0$  and  $\langle \mathbf{d}_i, \mathbf{d}_j \rangle = 0$ ,  $i \neq j$ , where  $\langle \cdot, \cdot \rangle$  is an inner product defined by  $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot A\mathbf{v}$ ,  $\forall \mathbf{u}, \mathbf{v} \in R^n$ .

[10 marks]

3. Explain why the Conjugate Gradient method is a direct method.

[3 marks]