DUBLIN CITY UNIVERSITY

SEMESTER ONE REPEAT EXAMINATION 2004

MODULE:	MS401M/MS401 Numerical Linear Algebra and Applics.
COURSE:	B. Sc. in Mathematical Sciences, B. Sc. in Financial and Actuarial Mathematics, M. Sc. in Financial and Industrial Mathematics.
YEAR:	4
EXAMINERS:	Mr. P. Cooper, Prof. M. H. A. Davies, Prof. T. Hurley, Dr. E. O'Riordan (ext. 5386).
TIME ALLOWED:	2 hours
INSTRUCTIONS:	Attempt any three questions. All questions carry equal marks.
REQUIREMENTS:	None

THE USE OF PROGRAMMABLE OR TEXT STORING CALCULATORS IS EXPRESSLY FORBIDDEN

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QUESTION 1

Let $\Omega = (0,1) \times (0,1)$ with $\partial\Omega = \overline{\Omega}\backslash\Omega$. Consider the problem

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = f(x,y), \quad (x,y) \in \Omega$$

$$u = g(x,y), \quad (x,y) \in \partial\Omega$$

which is approximated on the uniform mesh

$$\overline{\Omega}^N = \{(x_i, y_j) = (ih, jh) : i,j = 0, \ldots, N\}, \quad h = 1/N$$

by the finite difference scheme

$$L^N Z_{i,j} \equiv \delta_x^2 Z_{i,j} + \delta_y^2 Z_{i,j} + D_x^+ Z_{i,j} + D_y^+ Z_{i,j} - Z_{i,j} = f(x_i, y_j) \text{ on } \Omega^N,$$

$$Z_{i,j} = u(x_i, y_j) \text{ on } \partial\Omega^N.$$

(a) State and prove a comparison principle for the differential operator L. Hence, show that

$$\max_{\overline{\Omega}} |u| \leq \left(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f| \right).$$

[9 marks]

(b) Show that the difference operator L^N satisfies a corresponding discrete comparison principle. Hence, show that

$$\max_{\overline{\Omega}^N} |Z| \leq \left(\max_{\partial\Omega^N} |g| + \max_{\overline{\Omega}^N} |f| \right),$$

where Z is the solution of the finite difference scheme defined above.

[7 marks]

(c) Derive an expression for the truncation error and show that the solutions of the difference scheme converge (as $N \to \infty$) to the exact solution u.

[9 marks]

(d) Consider the special case of homogeneous boundary conditions, i.e., $u = 0$, for all $(x,y) \in \partial\Omega$. Using a natural ordering of the mesh points, write the difference scheme in the matrix form $A\vec{z} = \vec{b}$, where $\vec{z}$ is a vector containing the unknown internal nodal values. Write out explicitly the elements of A and $\vec{b}$.

[8 marks]

QUESTION 2

(a) Let $\overline{\Omega}^{N,M} = \{(x_i, t_j) | x_i = i/N,\ t_j = j/M\}_{i,j=0}^{N,M}$ be a uniform mesh. The operators D^+, D^- are, respectively, the forward and backward finite difference operators. Find an expression for the local truncation error for the finite difference scheme

$$D_t^+ U(x_i, t_j) + a D_x^- D_x^+ D_x^+ U(x_i, t_j) = f(x_i, t_j),\ (x_i, t_j) \in \Omega^{N,M}$$

as it applies to the third order partial differential equation

$$u_t + a u_{xxx} = f.$$

[13 marks]

(b) Consider the singularly perturbed parabolic problem: Find u such that

$$\begin{aligned} \varepsilon u_{xx} + a(x,t) u_x - u_t = f(x,t),\ (x,t) \in G = (0,1) \times (0,1] \\ u(0,t) = u(1,t) = 0,\ t \geq 0; \qquad u(x,0) = g(x), \quad 0 < x < 1 \\ 0 < \varepsilon \leq 1, \qquad a(x,t) \geq \alpha > 0,\ (x,t) \in \overline{G} \end{aligned}$$

and the following implicit finite difference scheme

$$\begin{aligned} \varepsilon \delta_x^2 U + a(x_i, t_j) D_x^- U - D_t^- U = f(x_i, t_j),\ (x_i, t_j) \in G^N \\ u(0, t_j) = u(1, t_j) = 0,\ t_j \geq 0; \qquad u(x_i, 0) = g(x_i), \quad 0 < x_i < 1 \\ \overline{G}^N = \{(x_i, t_j) | x_i = i/N,\ t_j = j/N\}_{i,j=0}^{N}. \end{aligned}$$

Write this difference scheme as a matrix equation of the form

$$A_j \vec{v}_j = \vec{v}_{j-1} + \vec{f}_j . \vec{v}_0 = \vec{g}$$

where $\vec{v}_j = (U(x_i, t_j))_{i=1}^{N-1}$. Write down the elements in the matrix A_j and the vector $\vec{f}_j$ explicitly. Impose a condition on ε, N, a so that the off–diagonal entries of the matrix A are non–negative. Propose an alternative implicit finite difference scheme for this problem, whose off-diagonal entries of the corresponding system matrix are non negative irrespective of ε. State a discrete comparison principle that this alternative difference operator will satisfy.

[20 marks]

QUESTION 3

(a) Given that A is a nonsingular matrix and x_k is an approximation to x, the exact solution of $Ax = b \neq 0$, show that for any compatible matrix and vector norms

$$\frac{1}{c(A)} \frac{\|Ax_k - b\|}{\|b\|} \leq \frac{\|x_k - x\|}{\|x\|} \leq c(A) \frac{\|Ax_k - b\|}{\|b\|}$$

where $c(A) = \|A\| \|A^{-1}\|$.

[9 marks]

(b) For a nonsingular block tridiagonal matrix A the eigenvalues λ of the SOR iteration matrix are related to the eigenvalues μ of the Jacobi iteration matrix by

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2$$

Deduce that the optimal value for ω is given by

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \mu_{max}^2}}$$

[9 marks]

(c) Given a square matrix A, a sequence of square matrices may be defined using

$$X_{k+1} = X_k + X_k(I - AX_k)$$

Show that

$$I - AX_k = (I - AX_0)^{2^k}.$$

[6 marks]

(d) Consider the system

$$S\mathbf{x} = T\mathbf{x} + \mathbf{b}$$

and suppose that S is nonsingular and that $\|S^{-1}T\| < 1$. Show that

$$\|\mathbf{x}\| \leq \frac{\|S^{-1}\mathbf{b}\|}{1 - \|S^{-1}T\|}$$

[9 marks]

QUESTION 4

Assume that A is a real symmetric positive definite matrix and $\mathbf{b}$ is an arbitrary vector. The Conjugate Gradient method for solving the equation $A\mathbf{x} = \mathbf{b}$ may be written in the following form:

Let $\mathbf{x}_0$ be given, and let $\mathbf{d}_0 = -\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$. For $k = 0, 1, 2, ...$, compute

$$\alpha_k = -\frac{\mathbf{r}_k \cdot \mathbf{d}_k}{\mathbf{d}_k \cdot A\mathbf{d_k}}, \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \quad \text{and} \quad \mathbf{r}_{k+1} = A\mathbf{x}_{k+1} - \mathbf{b} \ ;$$

stop if $\mathbf{r}_{k+1}$ is sufficiently small; otherwise define

$$\beta_k = \frac{\mathbf{r}_{k+1} \cdot A\mathbf{d}_k}{\mathbf{d}_k \cdot A\mathbf{d}_k}, \quad \mathbf{d}_{k+1} = -\mathbf{r}_{k+1} + \beta_k \mathbf{d}_k$$

and continue.

(a) For the Conjugate Gradient method, show that the subspace spanned by the search directions is identical to the subspace spanned by the residuals, which in turn is identical to the Krylov subspace $[\vec{r}_0, A\vec{r}_0, ..., A^k\vec{r}_0]$ for all $\alpha_j \neq 0, \ j \leq k$.

[12 marks]

(b) Show that $\vec{r}_i \cdot \vec{r}_j = 0$ and $< \vec{d}_i, \vec{d}_j >= 0, \ i \neq j \leq k, \ \alpha_j \neq 0, \ j \leq k,$ where $< \cdot, \cdot >$ is an inner product defined by $< \vec{u}, \vec{v} >= \vec{u} \cdot A\vec{v}, \quad \forall \vec{u}, \vec{v} \in R^n$.

[13 marks]

(c) Use the Conjugate Gradient method to solve the linear system

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

using $(x_1, x_2, x_3) = (0, 0, 0)$ as an initial approximation.

[8 marks]