Destabilisation of Functional Differential Equations by Noise

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Abstract

This paper extends, by an alternative method, a result of Mao (Systems and Control Letters, 1994) which shows that solutions of nonlinear differential equations can be destabilised by noise. Here, we show that a linear noise can always destabilise a general even-dimensional functional differential equation, with bounded or unbounded delay, and illustrate the general results for linear problems.

Key words: stochastic destabilisation, stochastic functional differential equation, Itô-Volterra equation, Volterra equation, Liapunov exponent, nonlinear system.
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1 Introduction

This paper studies the stability of solutions of functional differential equations which are perturbed by noise. In particular, we consider even-dimensional systems where the underlying deterministic system may be stable, but with the addition of a sufficiently strong *multiplicative* noise with the right configuration, the solution explodes exponentially fast. Observe that such a noise perturbation preserves the equilibrium of the underlying deterministic system. We do not concern ourselves with the destabilisation of the equilibrium solution by *additive* noise here, which is not equilibrium-preserving.

The paper deeply relates to two papers which study the stabilising and destabilising effects of noise on non-linear differential equations and functional differential equations. In Mao [1], it is shown that a large class of finite dimensional differential equations can be stabilised or destabilised by Brownian motion. Some of these results are extended in Appleby [2] to show that finite

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dimensional functional differential equations can be stabilised by noise, for sufficiently small delay. Thus this paper is a companion of the latter work. In [1], it is shown that a "skew" linear diffusion term will induce instability, and it is that result which we establish here. We remark, however, that this paper considers an *example* of a noise configuration which is destabilising, while [1] deals with a *class* of potentially destabilising linear noise perturbations.

The proof of instability differs substantially between [1] and this note. An important property which the solutions of stochastic differential equations (SDEs) which have an equilibrium at 0 satisfy— but which is not necessarily satisfied by solutions of stochastic functional differential equations— is that SDEs with zero equilibria are non-zero for all time, provided that the solution is initially non-zero. The line of proof in [1] relies on this fact. Instead, we adopt the partitioning approach of [2], showing that the nowhere differentiable solution of the Itô-Volterra equation can be completely characterised in terms of a linear stochastic differential equation— whose asymptotic behaviour is well-understood— and the smooth solution of a stochastic evolution, which can be analysed on a pathwise basis in the same manner as a deterministic functional differential equation.

2 Preliminaries

We first fix some standard notation. As usual, let $x \vee y$ denote the maximum of $x, y \in \mathbb{R}$. Denote by $C(I;J)$ the space of continuous functions taking the finite dimensional Banach space I onto the finite dimensional Banach space J. Let d be a positive integer. Let $(\mathbb{R}^d)^n$ be the n-fold Cartesian product of $\mathbb{R}^d$ with itself. Let $M_{d,d}(\mathbb{R})$ denote the space of all $d \times d$ matrices with real entries, and $C(\mathbb{R}^+; M_{d,d}(\mathbb{R}))$ stand for all continuous $d \times d$ matrix-valued functions with domain $\mathbb{R}^+$. Further denote by I_d the identity matrix in $M_{d,d}(\mathbb{R})$. Let $\|x\|$ stand for the Euclidean norm of $x \in \mathbb{R}^d$. If $A = (A_{ij}) \in M_{d,d}(\mathbb{R})$, it has norm $\|A\| = \max_{\|x\|=1} \|Ax\|$. Let the set of bounded continuous functions taking I onto J be $BC(I;J)$. For I a subinterval of $\mathbb{R}$, we define the sup norm of $\psi \in BC(I;\mathbb{R}^d)$ by $\|\psi\|_I = \sup_{t\in I} \|\psi(t)\|$. For $x \in C(I;\mathbb{R}^d)$, we denote the history of x up to time $t \geq 0$ by $x_t = \{x(s) \,:\, s \leq t\}$.

In this paper, we consider stochastic perturbations of the deterministic functional differential equation

$$x'(t) = f_1(t, x(t), x(t-\tau_1(t), \ldots, x(t-\tau_n(t))) + \int_{t-\tau_0(t)}^{t} f_2(s,t,x(s))\,ds, \quad t \geq 0 \quad (1)$$

where $n \in \mathbb{N}$. We make the following hypotheses on the structure of (1). Let

$f_1 \in C(\mathbb{R}^+ \times (\mathbb{R}^d)^{n+1}; \mathbb{R}^d)$, $f_2 \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d)$. Suppose moreover that f_1 satisfies global Lipschitz conditions in the space coordinates x, and satisfy global linear bounds in the space coordinates in that there exist non-negative constants K_j $j = 1, \ldots, n+1$ such that

$$\|f_1(t, x_1, \ldots, x_{n+1})\| \leq \sum_{j=1}^{n+1} K_j \|x_j\| \quad \text{for all } t \geq 0, x_j \in \mathbb{R}^d. \tag{2}$$

As for f_2, suppose there is a measurable function $k : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f_2(s, t, x) - f_2(s, t, y)\| \leq k(s, t)\|x - y\|, \tag{3}$$

where

$$\sup_{t \geq 0} \int_{t-\tau_0(t)}^{t} k(s, t) =: K_0 < \infty, \tag{4}$$

and

$$k \text{ bounded, and } f_2(s, t, 0) = 0 \quad \text{for all } s \leq t. \tag{5}$$

We now place some hypotheses on the delay functions τ_j, $j = 0, 1, \ldots, n$. Let $\tau_j \in C(\mathbb{R}^+; \mathbb{R}^+)$. We will study both problems with bounded delay, where

$$\overline{\tau} = \sup_{t \geq 0} \max_{j=0,\ldots,n} \tau_j(t) < \infty, \tag{6}$$

and also problems with unbounded delay in which there exists at least one $j = 1, \ldots, n$ such that

$$\sup_{t \geq 0} t - \tau_j(t) = \infty \tag{7}$$

or $\sup_{t \geq 0} t - \tau_0(t) = \infty$. In the case where

$$\lim_{t \to \infty} t - \tau_j(t) = \infty, \quad j = 0, 1, \ldots, n \tag{8}$$

the problem is said to have fading memory. These classifications are distinguished in more detail in Kolmanovskii and Myshkis [3].

We will study problems when $\overline{\tau}$ in (6) above can be infinite. We consider initial functions $\psi \in BC([-\overline{\tau}, 0]; \mathbb{R}^d)$, and suppose that

$$x(t) = \psi(t), \quad -\overline{\tau} \leq t \leq 0. \tag{9}$$

The Lipschitz continuity on f_1, f_2 (conditions (2), (3), (4), (5)), together with the continuity of the delay functions and one of the conditions specifying whether the problem is a bounded or unbounded delay (or fading memory) problem guarantee uniqueness of solutions of (1) subject to the initial function (9). As $f_1(t, 0, 0, \ldots, 0) = 0$, and $f_2(s, t, 0) = 0$ for all $s \leq t$, we observe that $x(t) = 0$ for all $t \geq 0$ is the unique solution whenever $\psi(t) = 0$ for all $t \in [-\overline{\tau}, 0]$.

In the sequel, we wish to study the instability of solutions of a stochastic version of (1), so it is natural to specialise the space of initial functions to be

$$C_\delta = \{\psi \in BC([-\overline{\tau}, 0]; \mathbb{R}^d) \; : \; \|\psi(t)\|_{[-\tau,0]} < \delta\} \tag{10}$$

for $\delta > 0$. Let $(B(t))_{t\geq 0}$ be a one-dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t = \sigma(B(s) \; : \; 0 \leq s \leq t)$. Next, let σ be a real constant and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11}$$

Let $m \in \mathbb{N}$, and $d = 2m$. Define $\Sigma \in M_{d\times d}(\mathbb{R})$ by the block diagonal matrix

$$\Sigma = \sigma \text{ diag } (J, J, \ldots, J). \tag{12}$$

We will study the stochastic functional differential equation

$$\begin{aligned} dX(t) = \Big(& f_1(t, X(t), X(t-\tau_1(t)), \ldots, X(t-\tau_n(t))) \\ & + \int_{t-\tau_0(t)}^{t} f_2(s, t, X(s))\, ds \Big) \, dt + \Sigma X(t)\, dB(t), \quad t \geq 0 \qquad \text{(13a)} \\ X(t) = {} & \psi(t), \quad -\overline{\tau} \leq t \leq 0. \qquad \text{(13b)} \end{aligned}$$

If the above same hypotheses on ψ, f_1, f_2 hold as before, then (13) has a unique continuous strong solution on $\mathbb{R}^+$ (see Mao [4], or Berger and Mizel [5], for instance). We shall denote solutions of (13) with initial function (13b) by $X(t, \psi)$. Once again, note that $X(t, 0) = 0$ for all $t \geq 0$. It is the stability of this zero solution we will study in this note.

2.1 Stochastic stability

If $\sigma = 0$, (13) collapses to the deterministic problem (1). Thus, by showing for some non-zero σ that the zero solution of (13) is almost surely unstable, we demonstrate that the class of even-dimensional functional differential equations above can be destabilised by noise. To this end, we consider a natural extension of exponential instability for stochastic differential equations (see Mao [4]), which is in the same spirit as the definition of instability of solutions of deterministic Volterra equations (see Driver [6], or Burton and Mahfoud [7]). We say the zero solution of (13) is *almost surely exponentially asymptotically unstable* if there is a $\delta > 0$, and a $\psi \in C_\delta$ such that

$$\liminf_{t\to\infty} \frac{1}{t} \log \|X(t, \psi)\| \geq \alpha \quad \text{a.s.}$$

for some $\alpha > 0$. For $\xi \in \mathbb{R}^d$ satisfying $\mathbb{E}[\|\xi\|^2] < \infty$, we also study the instability of the zero solution of the Itô-Volterra initial value problem

$$dX(t) = \left(f_1(t, X(t)) + \int_0^t f_2(s, t, X(s))ds\right)dt + \Sigma X(t)dB(t), \; t \geq 0, \tag{14a}$$
$$X(0) = \xi. \tag{14b}$$

Denoting solutions of (14) by $X(t, \xi)$, we say the zero solution of (14a) is *a.s. exponentially asymptotically unstable* if there is $\xi \neq 0$ a.s., such that

$$\liminf_{t\to\infty} \frac{1}{t} \log \|X(t, \xi)\| \geq \alpha \quad \text{a.s.}$$

for some $\alpha > 0$.

2.2 Organisation of the note

The following section comprises the main result of the paper, namely that the zero solution of (13) is almost surely exponentially asymptotically unstable for some sufficiently large value of σ^2. The results are then used to show that all non-zero initial conditions of the initial value Itô-Volterra equation give rise to exponentially exploding solutions for sufficiently large σ^2. Section 4 considers some specific examples of deterministic linear delay-differential equations and Volterra equations which have stable (or even uniformly, or exponentially asymptotically stable) zero solutions, for which the corresponding random problem can be made a.s. unstable, even though its mean value is exactly the same as the solution of the deterministic problem. In Section 5, we show that the method of proof which establishes that the zero solution of (13) is almost surely exponentially asymptotically unstable for sufficiently large value of σ^2 also applies to a general even-dimensional stochastic functional differential equation The proof of a technical result, required in Section 3, is relegated to Section 6.

3 Main Result

Our proofs depend on properties of some auxiliary stochastic processes, which we first introduce. Define the $M_{d,d}(\mathbb{R})$–valued stochastic process $(\Phi(t))_{t\geq 0}$ by $\Phi(0) = I_d$ and let it solve the matrix stochastic differential equation $d\Phi(t) = \Sigma\Phi(t)\, dB(t)$, where Σ is given by (12). Also introduce the $M_{2,2}(\mathbb{R})$–valued stochastic process $(\varphi(t))_{t\geq 0}$ by $\varphi(0) = I_2$ and let it solve the matrix stochastic differential equation $d\varphi(t) = \sigma J\varphi(t)\, dB(t)$, where J is given by (11).

Evidently, Φ is related to φ according to $\Phi(t) = \text{diag}\,(\varphi(t), \varphi(t), \ldots, \varphi(t))$. Next, define $\theta(t)$ for $t \geq 0$ by

$$\theta(t) = \begin{pmatrix} \cos(\sigma B(t)) & \sin(\sigma B(t)) \\ -\sin(\sigma B(t)) & \cos(\sigma B(t)) \end{pmatrix}, \tag{15}$$

so that

$$\varphi(t) = e^{\frac{\sigma^2}{2}t}\theta(t). \tag{16}$$

As $\theta(t)$ is invertible for all $t \geq 0$, so is $\Phi(t)$. By defining the $M_{d,d}(\mathbb{R})$–valued stochastic process $(\Theta(t))_{t\in\mathbb{R}}$ by $\Theta(t) = I_d$ for $t \leq 0$ and

$$\Theta(t) = \text{diag}\,(\theta(t), \theta(t), \ldots, \theta(t)), \quad t \geq 0 \tag{17}$$

and noting that $\Theta(t)^{-1} = \text{diag}\,(\theta(t)^{-1}, \theta(t)^{-1}, \ldots, \theta(t)^{-1})$, we have

$$\Phi(t) = e^{\frac{\sigma^2}{2}t}\Theta(t), \quad \Phi(t)^{-1} = e^{-\frac{\sigma^2}{2}t}\Theta(t)^{-1}. \tag{18}$$

We are now in a position to prove our main results.

Theorem 1 *Let $\sigma^2 > 2\sum_{j=0}^{n+1} K_j$. Then there exists $\psi \in BC([-\overline{\tau}, 0]; \mathbb{R}^d)$ such that the solution of (13), (13b) satisfies*

$$\|X(t)\| \geq \|\psi\|_{[-\overline{\tau},0]} e^{(\frac{\sigma^2}{2} - \sum_{j=0}^{n+1} K_j)t}, \quad t \geq 0, \ a.s.$$

PROOF. For $t \geq 0$, define the functional $R(t, X_t)$ by

$$R(t, X_t) = f_1(t, X(t), X(t - \tau_1(t)), \ldots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^{t} f_2(s, t, X(s))\, ds. \tag{19}$$

Viewing $R(t) = R(t, X_t)$, note that $t \mapsto R(t)$ is a continuous function on $\mathbb{R}^+$. Using (stochastic) integration by parts, we see that X given by (13), (13b) satisfies

$$X(t) = \Phi(t)\left(\psi(0) + \int_0^t \Phi(s)^{-1} R(s, X_s)\, ds\right). \tag{20}$$

Define $u(t) = \psi(t)$ for $t \in [-\overline{\tau}, 0]$, and $u(t) = \Theta(t)^{-1}X(t)$ for $t \geq 0$. By construction of Θ, one obtains

$$X(t) = \Theta(t)u(t), \quad \|X(t)\| = \|u(t)\|, \quad t \geq -\overline{\tau}. \tag{21}$$

Defining $y(t) = \Phi(t)^{-1}X(t)$ for $t \geq 0$ yields $y'(t) = \Phi(t)^{-1}R(t, X_t)$, using (20). By (18), $u(t) = e^{\frac{\sigma^2}{2}t}y(t)$ for $t \geq 0$, so

$$u'(t) = \frac{\sigma^2}{2}u(t) + F(t, u_t), \quad t \geq 0, \tag{22}$$

where $F(t,u_t) = \Theta(t)^{-1}R(t,(\Theta u)_t)$. To bound F, $\|\Theta(t)\| = \|\Theta(t)^{-1}\| = 1$ for $t \geq -\overline{\tau}$, (2), (3), (4), (5), and (18) give

$$\begin{aligned}
&\|F(t,u_t)\| \\
&\leq \|f_1(t,\Theta(t)u(t),\dots,\Theta(t-\tau_n(t))u(t-\tau_n(t))\| \\
&\qquad + \int_{t-\tau_0(t)}^{t} \|f_2(s,t,\Theta(s)u(s))\|\,ds \\
&\leq K_1\|u(t)\| + \sum_{j=2}^{n+1} K_j\|u(t-\tau_{j-1}(t))\| + \int_{t-\tau_0(t)}^{t} k(s,t)\|u(s)\|\,ds. \qquad (23)
\end{aligned}$$

Now, we choose $\beta \in [0, \frac{\sigma^2}{2} - \sum_{j=0}^{n+1} K_j)$, and, for $t \geq -\overline{\tau}$, define $v(t) = e^{-2\beta t}\|u(t)\|^2$. Consider the space of initial functions $\psi(\not\equiv 0)$ in C_δ denoted by $\Psi_\beta = \{\psi \in C_\delta \,:\, \mu : [-\overline{\tau},0] \to \mathbb{R}^+ : t \mapsto \mu(t) := e^{-2\beta t}\|\psi(t)\|^2 \text{ is increasing}\}$. Hence, if $\psi \in \Psi_\beta$, then $v : [-\overline{\tau},0] \to \mathbb{R}^+$ is an increasing function with $v(0) > 0$. By (22) and the definition of v we get

$$v'(t) = (\sigma^2 - 2\beta)v(t) + 2e^{-2\beta t}\langle u(t), F(t,u_t)\rangle, \qquad (24)$$

while (4), (23) in conjunction with the Cauchy-Schwarz inequality, the inequality $2xy \leq x^2 + y^2$ (for $x, y \geq 0$) and $\tau_j(t) \geq 0$ yields

$$\begin{aligned}
&2e^{-2\beta t}|\langle u(t), F(t,u_t)\rangle| \\
&\leq 2e^{-2\beta t}\|u(t)\|\Bigg(K_1\|u(t)\| + \sum_{j=2}^{n+1} K_j\|u(t-\tau_{j-1}(t))\| \\
&\qquad + \int_{t-\tau_0(t)}^{t} k(s,t)\|u(s)\|\,ds\Bigg) \\
&\leq K_1 e^{-2\beta t}\|u(t)\|^2 + \sum_{j=2}^{n+1} K_j e^{-2\beta t}(\|u(t)\|^2 + \|u(t-\tau_{j-1}(t))\|^2) \\
&\qquad + \int_{t-\tau_0(t)}^{t} k(s,t)e^{-2\beta t}(\|u(t)\|^2 + \|u(s)\|^2)\,ds \\
&\leq (2K_1 + \sum_{j=2}^{n+1} K_j + K_0)v(t) + \sum_{j=2}^{n+1} K_j e^{-2\beta\tau_{j-1}(t)} v(t-\tau_{j-1}(t)) \\
&\qquad + \int_{t-\tau_0(t)}^{t} k(s,t)e^{-2\beta(t-s)}v(s)\,ds.
\end{aligned}$$

As $\beta \geq 0$, (24) and the above bound give

$$\begin{aligned}
v'(t) \geq{}& (\sigma^2 - 2\beta - 2K_1 - \sum_{j=2}^{n+1} K_j - K_0)v(t) \\
&- \sum_{j=1}^{n+1} K_j v(t-\tau_{j-1}(t)) - \int_{t-\tau_0(t)}^{t} k(s,t)v(s)\,ds, \quad t \geq 0, \qquad (25)
\end{aligned}$$

while

$$v(0) > 0, \quad t \mapsto v(t) \text{ is increasing on } [-\overline{\tau}, 0], \text{ and } v \in C^1(\mathbb{R}^+; \mathbb{R}^+). \tag{26}$$

Using Lemma 6, we have

$$v(t) \geq v(0) e^{(\sigma^2 - 2\beta - 2\sum_{j=0}^{n+1} K_j)t},$$

so the definition of v yields $\|u(t)\| \geq \|u(0)\| e^{(\frac{\sigma^2}{2} - \sum_{j=0}^{n+1} K_j)t}$, $t \geq 0$. Thus, (21) yields

$$\|X(t)\| \geq \|X(0)\| e^{(\frac{\sigma^2}{2} - \sum_{j=0}^{n+1} K_j)t}. \tag{27}$$

Choosing $\psi \in \Psi_0$ now proves the result. □

We now turn to some special cases and variants of Theorem 1. If $K_0 = 0$, $K_j = 0$, $j = 2, \ldots, n+1$, then (13), (13b) reduce to

$$dX(t) = f_1(t, X(t))\, dt + \Sigma X(t)\, dB(t), \tag{28}$$

where f_1 satisfies $\|f_1(t, x)\| \leq K_1 \|x\|$. The following corollary of Theorem 1 is immediately evident.

Corollary 2 *Let $\sigma^2 > 2K_1$. Then the solution of (28) with $X(0) = \xi$ satisfies*

$$\|X(t, \xi)\| \geq \|\xi\| e^{(\frac{\sigma^2}{2} - K_1)t}, \quad t \geq 0, \ a.s.$$

This result slightly sharpens one of the examples of Theorem 4.1 in Mao [1].

We also consider the case of the Itô-Volterra initial value problem, with $K_j = 0$ for $j = 2, \ldots, n+1$, and $\tau_0(t) = t$, so that (13) reduces to (14a) with initial condition $X(0) = \xi$. A trivial reworking of the argument of Theorem 1 now yields the following Theorem.

Theorem 3 *Let $\sigma^2 > 2K_0 + 2K_1$. Then the solution of (14b) with initial condition $X(0) = \xi$ satisfies*

$$\|X(t, \xi)\| \geq \|\xi\| e^{(\frac{\sigma^2}{2} - K_1)t}, \quad t \geq 0, \ a.s.,$$

In this case, all solutions of (14b) with non-zero initial conditions are exponentially asymptotically unstable.

Remark 4

It may be observed that (27) represents a stronger result than stated in the hypothesis of Theorem 1: namely, if the initial function ψ belongs to Ψ_β defined above (for $\beta \in [0, \sigma^2/2 - \sum_{j=0}^{n+1} K_j))$, then the solution of (13) is exponentially unstable. However, for $\overline{\tau} = \infty$, $\beta \neq 0$, there is no interesting relation between the sup norm of the initial function and the initial value $X(0)$. Since the stability of solutions of Volterra equations is defined in terms of the norm of the initial history, the result of the Theorem, although weaker, is more in the spirit of the definition.

4 Linear Equations

To exemplify the above theory, suppose A and C are bounded and consider the linear delay differential equation and its stochastic analogue

$$x'(t) = A(t)x(t) + C(t)x(t-\tau), \tag{29}$$
$$dX(t) = (A(t)X(t) + C(t)X(t-\tau))\,dt + \Sigma X(t)\,dB(t). \tag{30}$$

If $X(t) = x(t) = \psi(t)$ for $t \in [-\tau, 0]$, then $\mathbb{E}[X(t)] = x(t)$, for all $t \geq 0$. Thus, the two systems share the same mean value. However, with $\sigma^2 > 2(\sup_{t\geq 0} \|A(t)\| + \sup_{t\geq 0} \|C(t)\|)$, the zero solution of (30) is a.s. exponentially asymptotically unstable, whether the zero solution of (29) is stable or not.

The same analysis holds true for the deterministic Volterra equation and its stochastic counterpart

$$x'(t) = A(t)x(t) + \int_0^t K(s,t)x(s)\,ds \tag{31}$$
$$dX(t) = (A(t)X(t) + \int_0^t K(s,t)X(s)\,ds)\,dt + \Sigma X(t)\,dB(t), \tag{32}$$

where A is bounded and $\sup_{t\geq 0} \int_0^t \|K(s,t)\| ds = K_0$ is finite. In this case, the initial value problem (32) is almost surely exponentially unstable for $\sigma^2 > 2\sup_{t\geq 0} \|A(t)\| + 2K_0$. Naturally, conditions on A and C (resp. K) can be imposed under which all solutions of (29), (resp. (31)) converge to zero as $t \to \infty$, or for which the zero solution is stable. There is a wealth of literature on this subject, which is reviewed extensively in, for instance, [8,9,3,10] and the references quoted therein.

5 Instability of Stochastic Functional Differential Equations

In this section, we study the noise-induced instability of the stochastic functional differential equation

$$dX(t) = f(t, X_t)\,dt + \Sigma X(t)\,dB(t), \tag{33a}$$
$$X(t) = \psi(t), \quad t \in [-\overline{\tau}, 0]. \tag{33b}$$

In (33), $\Sigma \in M_{d,d}(\mathbb{R})$ is the matrix defined in (12), $(B(t))_{t\geq 0}$ is standard Brownian motion. Further, $f \in C(\mathbb{R}^+; C([-\overline{\tau}, 0]; \mathbb{R}^d))$ satisfies $f(t, 0) = 0$ for all $t \geq 0$ and obeys the global linear bound

$$\|f(t, \phi)\| \leq K \sup_{-\overline{\tau}\leq s\leq t} \|\phi(s)\|, \tag{34}$$

for some $K > 0$ and all $(t, \phi) \in \mathbb{R}^+ \times C([-\overline{\tau}, 0]; \mathbb{R}^d)$. The initial function ψ is in $C([-\overline{\tau}, 0]; \mathbb{R}^d)$. Suppose also that f satisfies a local Lipschitz condition of the following form: for all $n \in \mathbb{N}$ there exists $K_n > 0$ such that if

$$\sup_{-\overline{\tau}\leq s\leq t} \|\phi_1(s)\| \vee \sup_{-\overline{\tau}\leq s\leq t} \|\phi_2(s)\| \leq n$$

then

$$\|f(t, \phi_{1t}) - f(t, \phi_{2t}\| \leq K_n \sup_{-\overline{\tau}\leq s\leq t} \|\phi_1(s) - \phi_2(s)\|, \quad t \geq 0. \tag{35}$$

Under conditions (34), (35), the stochastic functional differential equation (33) has a unique strong solution, whose value at time $t \geq 0$ we denote by $X(t, \psi)$. Note that $X(t, 0) = 0$ for all $t \geq 0$; this solution is called the zero solution of (33). Proceeding as for the Volterra equation (13) in Theorem 1, we will prove the following result.

Theorem 5 *Suppose that f satisfies (34), (35), and Σ satisfies (12). If $\sigma^2 > 2K$, there exists $\psi \in BC([-\overline{\tau}, 0]; \mathbb{R}^d)$ such that the solution of (33) obeys*

$$\|X(t, \psi)\| \geq \|\psi\|_{[-\overline{\tau},0]} e^{(\frac{\sigma^2}{2} - K)t}, \quad t \geq 0, \text{ a.s.}$$

PROOF. Define Φ, Θ, φ, θ as in (15)–(18), and let $u(t) = \Theta(t)^{-1}X(t)$ and $y(t) = \Phi(t)^{-1}X(t)$ as before. Then we have $u(t) = e^{\frac{\sigma^2}{2}t}y(t)$, $y'(t) = \Phi(t)^{-1}f(t, X_t)$ for $t \geq 0$. Hence u obeys (22), where

$$F(t, u_t) = \Theta(t)^{-1}f(t, (\Theta u)_t), \quad t \geq 0. \tag{36}$$

Using (34) and the fact that $\|\Theta(t)\| = \|\Theta(t)^{-1}\| = 1$, we have

$$\|F(t, u_t)\| \leq K \sup_{-\overline{\tau}\leq s\leq t} \|u(s)\|. \tag{37}$$

As in Theorem 1, by defining $v(t) = e^{-2\beta t}\|u(t)\|^2$ for $\beta \in [0, \sigma^2/2 - K)$, we see that v obeys (24). The bound on $2e^{-2\beta t}|\langle u(t), F(t, u_t)\rangle|$ required proceeds as in Theorem 1, where we bound $\|F(t, u_t)\|$ using (37). This yields

$$2e^{-2\beta t}|\langle u(t), F(t, u_t)\rangle| \le Kv(t) + K \sup_{-\overline{\tau} \le s \le t} v(s). \tag{38}$$

Suppose that $\psi(\not\equiv 0)$ is in the space Ψ_β defined in Theorem 1. Then by (24), (38), v satisfies

$$v'(t) \ge (\sigma^2 - 2\beta - K)v(t) - K \sup_{-\overline{\tau} \le s \le t} v(s), \quad t \ge 0 \tag{39}$$

and (26). It is not difficult to adapt the line of proof of Lemma 6 below to show that v obeying (26) and (39) satisfies $v'(t) \ge (\sigma^2 - 2\beta - 2K)v(t)$, for $t \ge 0$. Hence, the positivity of v yields $v(t) \ge v(0)\exp\{(\sigma^2 - 2\beta - 2K)t\}$, for all $t \ge 0$. The proof concludes as in Theorem 1 by choosing $\psi \in \Psi_0$. $\square$

6 Proof of Lemma 6

We now prove a result used in the proof of Theorem 1.

Lemma 6 *If v satisfies (25), (26), where k satisfies (5), then*

$$v(t) \ge v(0)e^{(\sigma^2 - 2\beta - 2\sum_{j=0}^{n+1} K_j)t}, \quad t \ge 0.$$

PROOF. Note that the monotonicity of v on $[-\overline{\tau}, 0]$ yields

$$\sum_{j=2}^{n+1} K_j v(t - \tau_{j-1}(t)) \le \sum_{j=2}^{n+1} K_j v(t), \tag{40}$$

$$\int_{t-\tau_0(t)}^{t} k(s,t)v(s)\,ds \le \int_{t-\tau_0(t)}^{t} k(s,t)v(t)\,ds \le K_0 v(t), \tag{41}$$

for $t = 0$. Hence $v'(0) \ge (\sigma^2 - 2\beta - 2\sum_{j=0}^{n+1} K_j)v(0) > 0$. Since $t \mapsto v(t)$ is in $C^1(\mathbb{R}^+)$, there is a $t_0 > 0$ such that $v'(t) > 0$ for all $t \in [0, t_0)$, and $t_0 = \inf\{t > 0 \,:\, v'(t) = 0\}$. Thus $v(t_0) > 0$ and v is increasing on $[-\overline{\tau}, t_1)$, so (40), (41) are true at $t = t_0$. This gives

$$0 = v'(t_0) \ge (\sigma^2 - 2\beta - 2\sum_{j=0}^{n+1} K_j)v(t_0) > 0,$$

a contradiction. Thus $v'(t) > 0$, and so (40), (41) are true for all t. Therefore

$$v'(t) \ge (\sigma^2 - 2\beta - 2\sum_{j=0}^{n+1} K_j)v(t).$$

The positivity of v now assures the result. □

Remark 7

A more general result than Lemma 6, but with a weaker conclusion is the "converse Halanay Inequality". See Theorem 4.2 in [11].

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