Stabilisation of Functional Differential Equations by Noise

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Abstract

This paper extends, by an alternative method, a result of Mao (Systems and Control Letters, 1994) which shows that solutions of nonlinear differential equations can be stabilised by noise. Here, we show that a linear multiplicative noise can always stabilise a general finite-dimensional functional differential equation, whenever the delay is sufficiently small. The result is also extended to a scalar functional differential equation with nonlinear multiplicative noise.

Key words: stochastic stabilisation, stochastic functional differential equation, almost sure exponential stability, Liapunov exponent, top Liapunov exponent, nonlinear system, Volterra equations
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1 Introduction

This paper is inspired by Mao's paper in this journal [1]. In it, he proves that general finite dimensional stochastic differential equations can be stabilised or destabilised by Brownian motion. As a consequence, one would expect that a similar result should be true for delay differential equations, or functional differential equations, if the delay is small. We show that this is true in respect of stabilisation in this note. Mao has made use of this idea before, in [2], or Chapter 7 of [3], for example.

The method of proof in [1] will not go through to the delay-dependent case in general, however, as it requires that the solution of the equation is always non-zero, almost surely. It has recently been shown in Appleby and Buckwar [4]

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that certain classes of linear stochastic delay differential equations have oscillatory solutions, even for arbitrarily small delay times, and in the case where the corresponding deterministic equation is nonoscillatory. Even in the presence of a solution which is always non-zero, a.s., it is difficult to use conventional techniques which might give a stabilisation result: for instance showing that the p^{th} moment of the solution converges exponentially, for $0 < p < 1$, (see Mao [5]).

Here we adopt a different approach. Observing that the (negative) contribution to the a.s. top Liapunov exponent is $-\sigma^2/2$ when there is a diffusion coefficient of the form σx, we write the solution as the product of a geometric Brownian motion and another stochastic process. By showing that the second process has top Liapunov exponent less than $\sigma^2/2$ when the delay time is sufficiently small, we can prove the desired result.

2 Preliminaries

We first fix some standard notation. Denote by $C(I;J)$ the space of continuous functions taking the finite dimensional Banach space I onto the finite dimensional Banach space J. Let d be a positive integer. Let $(\mathbb{R}^d)^n$ be the n-fold Cartesian product of $\mathbb{R}^d$ with itself. Let $M_{d,d}(\mathbb{R})$ denote the space of all $d \times d$ matrices with real entries, and $C(\mathbb{R}^+; M_{d,d}(\mathbb{R}))$ stand for all continuous $d \times d$ matrix-valued functions with domain $\mathbb{R}^+$. Further denote by I_d the identity matrix in $M_{d,d}(\mathbb{R})$. Let $\|x\|$ stand for the Euclidean norm of $x \in \mathbb{R}^d$. If $A = (A_{ij}) \in M_{d,d}(\mathbb{R})$, we will take as its norm $\|A\| = \max_{\|x\|=1} \|Ax\|$.

In this paper, we consider stochastic perturbations of the deterministic functional differential equation

$$x'(t) = f_1(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_n(t)) + \int_{t-\tau_0(t)}^{t} f_2(s,t,x(s))\,ds, \quad t \geq 0, \quad (1)$$

where $n \in \mathbb{N}$. We make the following hypotheses on the structure of this equation. Let $f_1 \in C(\mathbb{R}^+ \times (\mathbb{R}^d)^{n+1}; \mathbb{R}^d)$, $f_2 \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d)$. Suppose, moreover, that f_1 and f_2 satisfy the following global Lipschitz conditions in the space parameters x, so that for all $t \geq 0$ there exists $L > 0$ such that

$$\|f_1(t,x_1,\ldots,x_{n+1}) - f_1(t,y_1,\ldots,y_{n+1})\| \leq L \sum_{j=1}^{n+1} \|x_j - y_j\|, \; x_j, y_j \in \mathbb{R}^d, \quad (2)$$

$$\|f_2(s,t,x) - f_2(s,t,y)\| \leq L\|x-y\|, \quad 0 \leq s \leq t, x, y \in \mathbb{R}^d. \quad (3)$$

Moreover, f_1 and f_2 satisfy global linear bounds in that there exist nonnegative constants K_j, $j = 0, \dots n+1$ such that

$$\|f_1(t, x_1, x_2, \dots, x_{n+1})\| \le \sum_{j=1}^{n+1} K_j \|x_j\| \quad \text{for all } t \ge 0, x_j \in \mathbb{R}^d \tag{4}$$

$$\|f_2(s, t, x)\| \le K_0 \|x\| \quad \text{for all } 0 \le s \le t, x \in \mathbb{R}^d. \tag{5}$$

We now place some hypotheses on the delay functions τ_j, $j = 0, \dots, n$. Let $\tau_j \in C(\mathbb{R}^+; \mathbb{R}^+)$, so that $\overline{\tau}(t) = \max_{j=0,\dots,n} \tau_j(t)$ is a continuous function on $\mathbb{R}^+$. Suppose further that $\sup_{t \ge 0} \overline{\tau}(t) = \overline{\tau}$, i.e., there exists a finite $\overline{\tau}$ satisfying

$$\overline{\tau} = \sup_{t \ge 0} \max_{j=0,\dots,n} \tau_j(t). \tag{6}$$

Let $\phi \in C([-\overline{\tau}, 0], \mathbb{R}^d)$, where $\overline{\tau}$ satisfies (6). If

$$x(t) = \psi(t), \quad -\overline{\tau} \le t \le 0, \tag{7}$$

f_1, f_2 satisfy (2), (3), (4), (5) and τ_j satisfy the above continuity hypotheses, it then follows that there is a unique solution of (1). Note moreover that if $\psi(t) = 0$ for all $-\overline{\tau} \le t \le 0$, then $x \equiv 0$ is the unique solution of (1), which we call the zero solution of (1). We wish to study the almost sure asymptotic stability of this solution.

Towards this end, let $(B(t))_{t \ge 0}$ be a one dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{F}_t = \sigma(B(s) : 0 \le s \le t)$. Now, let σ be a real constant, and consider the stochastic functional differential equation

$$dX(t) = \Bigg(f_1(t, X(t), X(t - \tau_1(t)), \dots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^{t} f_2(s, t, X(s))\, ds \Bigg) dt + \sigma X(t)\, dB(t), \quad t \ge 0, \tag{8}$$

$$X(t) = \psi(t), \quad -\overline{\tau} \le t \le 0. \tag{9}$$

If above the assumptions on ψ, f_1, f_2 and τ_j, $j = 0, \dots, n$ hold, then (8) has a unique, strong continuous solution on $\mathbb{R}^+$. Again, if $\psi \equiv 0$, we have that $X(t) \equiv 0$ for all $t \ge 0$, a.s..

Note that when $\sigma = 0$, the problem collapses to the deterministic functional differential equation (1). Hence we ask whether there are values of $\sigma \ne 0$ such that $\lim_{t \to \infty} X(t) = 0$ a.s., or *a fortiori*, that there exists $\lambda > 0$ such that

$$\limsup_{t \to \infty} \frac{1}{t} \log \|X(t)\| \le -\lambda, \quad \text{a.s.} \tag{10}$$

The main result of this paper (in Section 3) shows that, when $\overline{\tau}$ defined by (6) is sufficiently small there is a range of values of σ can be chosen in (8) such that (10) holds for solutions of (8), regardless of the stability properties of the deterministic system (1). Some technical results which support this claim are relegated to the last section of the paper.

In Section 4, we specialise these results to linear systems, showing which the solutions of a deterministic system can explode, the equilibrium of the corresponding stochastic system can be a.s. exponentially stable, even though the mean value satisfies the deterministic problem.

In Section 5, we show that the result extends to a general scalar stochastic functional differential equation (SFDE) of the form $dX(t) = f(t, X_t)\,dt + \sigma h(X(t))\,dB(t)$ where h is a nonlinear function satisfying $\underline{h}|x| \le |h(x)| \le \overline{h}|x|$, and $f(t, \varphi) \in C([-\tau, 0]; \mathbb{R})$ is locally Lipschitz and globally linearly bounded. Again, if the delay time is sufficiently small the noise intensity σ is sufficiently large, the zero solution is a.s. exponentially asymptotically stable, even if the underlying deterministic problem $x'(t) = f(t, x_t)$ is unstable.

3 Linear multiplicative noise in finite dimensions

Before we start to prove our main result, we need some auxiliary functions. Suppose $\Phi(t) = I_d$, for $t \in [-\overline{\tau}, 0]$ and consider the matrix stochastic differential equation $d\Phi(t) = \sigma\Phi(t)\,dB(t)$. This has solution $\Phi(t) = \varphi(t)I_d$ where $(\varphi(t))_{t\ge 0}$ is the scalar geometric Brownian motion which satisfies $d\varphi(t) = \sigma\varphi(t)\,dB(t)$ with $\varphi(t) = 1$, $t \in [-\overline{\tau}, 0]$ and hence is given by

$$\varphi(t) = e^{-\frac{\sigma^2}{2}t + \sigma B(t)}. \tag{11}$$

Thus $\Phi(t)^{-1}$ exists for all $t \ge -\overline{\tau}$ a.s., and in particular $\Phi(t)^{-1} = \varphi(t)^{-1}I_d$, so $\|\Phi(t)\| = \varphi(t)$, $\|\Phi(t)^{-1}\| = \varphi(t)^{-1}$. For $t \ge 0$, let $R(t) = f_1(t, X(t), \dots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f_2(s, t, X(s))\,ds$, so that applying Itô's rule shows that X satisfying (8), (9) has representation

$$X(t) = \Phi(t)\left(\psi(0) + \int_0^t \Phi(s)^{-1}R(s)\,ds\right), \quad t \ge 0. \tag{12}$$

Next, let $y(t) = \Phi(t)^{-1}X(t)$ for $t \ge -\overline{\tau}$. As $t \mapsto \Phi(t), R(t)$ are continuous, (12) implies $y \in C^1(\mathbb{R}^+; \mathbb{R}^d)$. In fact, for $t \ge 0$, as $t - \tau_j(t) \ge -\overline{\tau}$, we get

$$y'(t) = \Phi(t)^{-1}\Bigg(f_1(t, \Phi(t)y(t), \dots, \Phi(t - \tau_n(t))y(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f_2(s, t, \Phi(s)y(s))\,ds\Bigg). \tag{13}$$

Thus

$$\|y'(t)\| \le K_1\|y(t)\| + \sum_{j=2}^{n+1} K_j\varphi(t)^{-1}\varphi(t-\tau_{j-1}(t))\|y(t-\tau_{j-1}(t))\| \\ + K_0\int_{t-\tau_0(t)}^t \varphi(t)^{-1}\varphi(s)\|y(s)\|\,ds,$$

and so by defining $p(t)$ for $t \ge \overline{\tau}$ by

$$p(t) = K_1 + \sum_{j=2}^{n+1} K_j\varphi(t)^{-1}\varphi(t-\tau_{j-1}(t)) + K_0\int_{t-\tau_0(t)}^t \varphi(t)^{-1}\varphi(s)\,ds, \tag{14}$$

we obtain, for $t \ge \overline{\tau}$

$$\|y(t)\| \le \|y(\tau)\| + \int_{\overline{\tau}}^t p(s) \max_{s-\overline{\tau}\le u\le s} \|y(u)\|\,ds.$$

Letting $y^*(t) = \max_{t-\overline{\tau}\le s\le t}\|y(s)\|$, for $t \ge \overline{\tau}$, we have $y^*(t) \le \|y(\overline{\tau})\| + \int_{\overline{\tau}}^t p(s)y^*(s)\,ds$. Gronwall's inequality now gives

$$\|y(t)\| \le y^*(t) \le \|y(\overline{\tau})\|e^{\int_{\overline{\tau}}^t p(s)\,ds}, \quad t \ge \overline{\tau}.$$

Suppose, now that there exists a finite deterministic constant C such that

$$\limsup_{t\to\infty} \frac{1}{t}\int_{2\overline{\tau}}^t p(s)\,ds \le C. \tag{15}$$

As $\lim_{t\to\infty}\frac{1}{t}\log\|\Phi(t)\| = -\frac{\sigma^2}{2}$, a.s., (15) and the definition of y gives

$$\limsup_{t\to\infty} \frac{1}{t}\log\|X(t)\| \le -\frac{\sigma^2}{2} + C, \quad \text{a.s.} \tag{16}$$

To show that (15) is true, and to obtain an explicit formula for C, we prove the following two lemmata, whose proofs are relegated to section 7.

Lemma 1 *Suppose $\tau \in C(\mathbb{R}^+;[0,\overline{\tau}])$, and $\varphi(t)$ satisfies (11). Then*

$$\limsup_{t\to\infty} \frac{1}{t}\int_{2\overline{\tau}}^t \varphi(s)^{-1}\varphi(s-\tau(s))\,ds \le e^{\sigma^2\overline{\tau}}, \quad a.s.,$$

Lemma 2 *Let $\tau \in C(\mathbb{R}^+;[0,\overline{\tau}])$ and $q_1(t) = \int_{t-\tau(t)}^t \varphi(t)^{-1}\varphi(s)\,ds$, where $\varphi(t)$ satisfies (11). Then*

$$\limsup_{t\to\infty} \frac{1}{t}\int_{2\overline{\tau}}^t q_1(s)\,ds \le e^{\sigma^2\overline{\tau}/2}\int_0^{\overline{\tau}} e^{\sigma^2 u/2}\,du, \quad a.s.$$

Applying Lemma 1 to each of the terms in the sum on the righthand side of (14), and applying Lemma 2 to the integral term yields (15) with $C =$

$K_1 + \sum_{j=2}^{n+1} K_j e^{\sigma^2\overline{\tau}} + K_0 e^{\sigma^2\overline{\tau}/2} \int_0^{\overline{\tau}} e^{\sigma^2 u/2}\,du$, by (14). Now, for $\overline{\tau} \geq 0$, define

$$\lambda_1(\sigma^2, \overline{\tau}) = \frac{\sigma^2}{2} - K_1 - \sum_{j=2}^{n+1} K_j e^{\sigma^2\overline{\tau}} - K_0 \frac{1}{\sigma^2/2}(e^{\sigma^2\overline{\tau}} - e^{\sigma^2\overline{\tau}/2}). \tag{17}$$

If $K_0 = 0$, $K_j = 0$ for $j = 2, \ldots, n+1$, there is no delay term in (8), which reduces to

$$dX(t) = f(t, X(t))\,dt + \sigma X(t)\,dB(t). \tag{18}$$

By (16) and the above, we have

Theorem 3 *Suppose that X solves (18), and there exists $K_1 > 0$ such that $\|f(t,x)\| \leq K_1\|x\|$. If $\sigma^2 > 2K_1$, then*

$$\limsup_{t\to\infty} \frac{1}{t} \log\|X(t)\| \leq -\left(\frac{\sigma^2}{2} - K_1\right), \quad a.s..$$

This is nothing but Theorem 3.2 in Mao [1]. But we are interested in extending stability to the delay equation, so assume now that a delay term is present in (8) by considering the case where at least one of K_0 or K_j, $j = 2, \ldots, n+1$ is positive. Define $\mathbf{K} = (K_0, K_1, \ldots K_{n+1})^T$. Then there exists a unique $\tau = \tau(\sigma^2, \mathbf{K}) > 0$ such that $\lambda_1(\sigma^2, \tau(\sigma^2, \mathbf{K})) = 0$ and $\lambda_1(\sigma^2, \tau) > 0$ for all $\tau < \tau(\sigma^2, \mathbf{K})$. In fact, we can explicitly compute

$$\tau(\sigma^2, \mathbf{K}) = \frac{2}{\sigma^2} \log\left(\frac{K_0 + \sqrt{K_0^2 + \sigma^2(\frac{1}{2}\sigma^2 - K_1)(2K_0 + \sigma^2\sum_{j=2}^{n+1} K_j)}}{2K_0 + \sigma^2\sum_{j=2}^{n+1} K_j}\right). \tag{19}$$

This gives us our main result.

Theorem 4 *Suppose that $\sigma^2 > 2\sum_{j=1}^{n+1} K_j$, and let $\tau(\sigma^2, \mathbf{K})$ be given by (19). If $\overline{\tau}$ defined by (6) satisfies $\overline{\tau} < \tau(\sigma^2, \mathbf{K})$, then the solution of (8) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log\|X(t)\| \leq -\lambda_1(\sigma^2, \overline{\tau}) < 0, \quad a.s.,$$

where λ_1 is defined by (17).

Thus, provided there is enough noise (i.e., $\sigma^2 > 2\sum_{j=1}^{n+1} K_j$), and the maximum delay ($\overline{\tau}$) is sufficiently small, a functional differential equation can be stabilised by noise, even if the original deterministic problem was unstable. We will address the issue of stabilisation directly in Section 4 for linear equations.

Another interesting question is: what is the maximum possible delay time that can be admitted in (1), so that the above analysis ensures that a $\sigma^2 > 0$ can be chosen to stabilise the equation. We answer this question in the next theorem.

Theorem 5 *Let $\tau(x,\mathbf{K})$ be given by (19). Then there exists a bounded continuous function $\overline{\tau} \in C(\mathbb{R}^{n+1};\mathbb{R})$ defined by $\overline{\tau}(\mathbf{K}) = \sup_{x>2\sum_{j=1}^{n+1} K_j} \tau(x,\mathbf{K})$, such that if $\overline{\tau}$ defined by (6) satisfies $\overline{\tau} < \overline{\tau}(\mathbf{K})$, then exists an open interval S, with $\inf S > 2\sum_{j=1}^{n+1} K_j$, such that if $\sigma^2 \in S$, then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| < 0, \quad a.s.,$$

where X solves (8).

PROOF. By the form of $\tau(\sigma^2,\mathbf{K})$ in (19), it is evident that $\lim_{x\to\infty} \tau(x,\mathbf{K}) = 0$ for fixed $\mathbf{K}$. Hence $\overline{\tau}(\mathbf{K})$ defined above is finite. As $\tau(\cdot,\mathbf{K})$ is continuous, and $\tau(2\sum_{j=1}^{n+1} K_j,\mathbf{K}) = 0$, there is a finite $\overline{\sigma}^2 > 2\sum_{j=1}^{n+1} K_j$ such that $\tau(\sigma^2,\mathbf{K}) \leq \tau(\overline{\sigma}^2,\mathbf{K}) = \overline{\tau}(\mathbf{K})$ for all $\sigma^2 > 2\sum_{j=1}^{n+1} K_j$. Now, let $\overline{\tau} < \overline{\tau}(\mathbf{K})$. Clearly, there exists an open interval S with $\overline{\sigma}^2 \in S$ such that for $\sigma^2 \in S$ we have $\overline{\tau} < \tau(\sigma^2,\mathbf{K})$. Applying Theorem 4 to these choices of σ^2 now suffices. □

A more explicit expression for $\overline{\tau}(\mathbf{K})$ can be obtained at the expense of some sharpness. For $\tau \geq 0$, define λ_2 by $\lambda_2(\sigma^2,\overline{\tau}) = \frac{\sigma^2}{2} - K_1 - \sum_{j=2}^{n+1} K_j e^{\sigma^2\overline{\tau}} - K_0\overline{\tau}e^{\sigma^2\overline{\tau}}$ and observe that $\lambda_1(\sigma^2,\overline{\tau}) > \lambda_2(\sigma^2,\overline{\tau})$, where λ_1 is given by (17), so that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| \leq -\lambda_2(\sigma^2,\overline{\tau}), \quad \text{a.s.}$$

Indeed, if $\sigma^2 = x > 2\sum_{j=1}^{n+1} K_j$, there is a well-defined function $\tau_2(x,\mathbf{K}) > 0$ (which is smooth in x) satisfying $\lambda_2(x,\tau_2(x,\mathbf{K})) = 0$, and for $\tau < \tau_2(x,\mathbf{K})$ obeys $\lambda_2(x,\tau) > 0$. The same line of argument as in Theorem 5, now gives

Theorem 6 *Let $\tau = \overline{\tau}(\mathbf{K}) > 0$ be the unique positive zero of*

$$e(K_0\tau + \sum_{j=2}^{n+1} K_j)e^{2K_1\tau} - \frac{1}{2\tau} = 0. \tag{20}$$

If $\overline{\tau}$ given by (6) satisfies $\overline{\tau} < \overline{\tau}(\mathbf{K})$, then there exists an open interval S satisfying $\inf S > 2\sum_{j=1}^{n+1} K_j$ and containing the point $\overline{\sigma}^2(\mathbf{K}) = 2K_1 + \frac{1}{\overline{\tau}(\mathbf{K})}$, such that for $\sigma^2 \in S$, $\limsup_{t\to\infty} \frac{1}{t}\log\|X(t)\| < 0$, a.s.

PROOF. Let $x > 2\sum_{j=1}^{n+1} K_j$. Since $\tau_2(x,\mathbf{K}) \leq \tau(x,\mathbf{K})$, (where τ is defined by (19)), $\lim_{x\to\infty} \tau_2(x,\mathbf{K}) = 0$. Since $\tau_2(2\sum_{j=1}^{n+1} K_j,\mathbf{K}) = 0$, $x \mapsto \tau_2(x,\mathbf{K})$ has at least one turning point. Next, note that τ_2 satisfies

$$(K_0\tau_2(x) + \sum_{j=2}^{n+1} K_j)e^{x\tau_2(x)} = \frac{x}{2} - K_1, \tag{21}$$

so that $\tau_2'(x)(K_0 e^{x\tau_2(x)} + x[x/2 - K_1]) = 1/2 - \tau_2(x)(x/2 - K_1)$. If $\tau_2'(x^*) = 0$, differentiating again shows $\tau_2''(x^*) < 0$, so $\tau_2(x, \mathbf{K})$ has a global maximum at $x = \overline{x}(\mathbf{K})$. Thus $\overline{\tau}(\mathbf{K}) = \tau_2(\overline{x}(\mathbf{K}), \mathbf{K}) = \frac{1}{\overline{x}(\mathbf{K}) - 2K_1}$, and $\overline{x}(\mathbf{K}) = 2K_1 + \frac{1}{\tau_2(\mathbf{K})}$. With $x = \overline{x}(\mathbf{K})$ in (21), using $\overline{\tau}(\mathbf{K}) = \frac{1}{\overline{x}(\mathbf{K}) - 2K_1}$ gives $\tau = \overline{\tau}(\mathbf{K})$ in (20). $\square$

4 Linear systems

We consider briefly the special case of linear equations. Consider, for simplicity the autonomous deterministic and stochastic systems

$$x'(t) = Ax(t) + Bx(t - \tau), \tag{22}$$
$$dX(t) = (AX(t) + BX(t - \tau))\, dt + \sigma X(t)\, dB(t) \tag{23}$$

which have the same initial conditions. It is immediate that $\mathbb{E}[X(t)] = x(t)$ for all $t \geq 0$. However, (23) can be stable even when (22) is not. By Theorem 5, if $\sigma^2 > 2(\|A\| + \|B\|)$, and $\tau < \sigma^{-2} \log((\sigma^2/2 - \|A\|)/\|B\|)$, then $\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| < 0$, a.s.. Indeed X has negative top Liapunov exponent for some $\sigma^2 > 2(\|A\| + \|B\|)$, whenever $\tau < \tau^*$, where $\tau^* > 0$ solves $1 - 2\tau^* \|B\| e e^{2\|A\|\tau^*} = 0$. On the other hand, (22) will be unstable if the transcendental equation $\det(\lambda I - A - Be^{-\lambda\tau}) = 0$ has a solution with $\Re e(\lambda) > 0$.

As a definite example of this phenomenon, consider the case where A, B are real numbers, and for simplicity, $B > 0$. The solution of (22) then explodes exponentially whenever $A > 0$, for instance. Hence the systems (22), (23) share the same mean value, but (23) is stable for some value of σ^2 provided $\tau < \tau^*$, where $\tau^* > 0$ solves $1 - 2\tau^* B e e^{2A\tau^*} = 0$.

5 Nonlinear multiplicative noise for a scalar SFDE

Let $h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous, and globally linearly bounded. Suppose further that $h(0) = 0$ and that h is continuously differentiable in some open interval containing the origin, with $h'(0) \neq 0$. We assume, without loss of generality in the sequel, that $h'(0) = 1$. Finally, suppose that there exist $0 < \underline{h} \leq 1 \leq \overline{h} < \infty$ such that

$$\underline{h}|x| \leq |h(x)| \leq \overline{h}|x|, \quad x \in \mathbb{R}. \tag{24}$$

Let $\tau > 0$. Suppose that $f : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ be a locally Lipschitz continuous functional which has the global linear bound

$$|f(t, \varphi)| \leq K\|\varphi\|, \quad (t, \varphi) \in \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}). \tag{25}$$

Here, for $\varphi \in C([-\tau,0];\mathbb{R})$ we have defined $\|\varphi\| = \max_{-\tau\le s\le 0}|\varphi(s)|$. Also, for $x \in C(\mathbb{R}^+;\mathbb{R})$ define $x_t = \{x(t+\theta) : \theta \in [-\tau,0]\}$. Hence there is a unique continuous strong solution to the stochastic functional differential equation

$$dX(t) = f(t,X_t)\,dt + \sigma h(X(t))\,dB(t), \tag{26a}$$
$$X(t) = \psi(t), \quad t \in [-\tau,0], \tag{26b}$$

whenever $\psi \in C([-\tau,0];\mathbb{R})$, and $B = \{B(t);\mathcal{F}^B_t; 0 \le t < \infty\}$ is a standard one-dimensional Brownian motion. Without loss, we suppose $\sigma > 0$. The hypotheses imply that if $X(t) = \psi(t) \equiv 0$ for $t \in [-\tau,0]$, then $X(t) = 0$ for all $t \ge 0$, a.s. This solution is called the zero solution of (26).

Theorem 7 *Under the hypotheses (24), (25), the solution of (26) satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log|X(t)| \le -\frac{1}{2}\sigma^2\underline{h}^2 + 2K\frac{\overline{h}^2}{\underline{h}^2}e^{\sigma^2\overline{h}^2\tau}, \quad a.s. \tag{27}$$

once $\psi(t) \not\equiv 0$ for $t \in [-\tau,0]$.

Before turning to the proof of this result, we remark that the stability of (26) can be analysed as in Section 3, and we can prove analogues of Theorems 4, 5. More precisely, if $\sigma^2 > 4K\overline{h}^2/\underline{h}^4$ then $\tau < 2\sigma^{-2}\overline{h}^{-2}\log(\sigma^2\underline{h}^2/4K\overline{h})$ suffices to give $\limsup_{t\to\infty}\frac{1}{t}\log|X(t)| < 0$, a.s. Indeed, if $\tau < \underline{h}^4/4eK\overline{h}^4$, there is an interval of values of $\sigma^2 > 4K\overline{h}^2/\underline{h}^4$ such that $\limsup_{t\to\infty}\frac{1}{t}\log|X(t)| < 0$, a.s.

To prove Theorem 7, Lemma 8 is required. Its proof is relegated to Section 7.

Lemma 8 *Let $\overline{\tau} > 0$, and suppose that $\tau : [\overline{\tau},\infty) \to [0,\overline{\tau}]$ is adapted to the filtration generated by the standard Brownian motion B. Then*

$$\limsup_{t\to\infty}\frac{1}{t}\int_{\overline{\tau}}^t e^{B(s)-B(s-\tau(s))}\,ds \le 2e^{\frac{1}{2}\overline{\tau}}, \quad a.s. \tag{28}$$

Proof (Theorem 7) Define $\tilde{h}(x) = \frac{h(x)}{x}$ for $x \ne 0$ and $\tilde{h}(0) = 1$. Then $\tilde{h}$ is a continuous and globally bounded function with $\underline{h} \le |\tilde{h}(x)| \le \overline{h}$. Next, define the process ϕ for $t \in [-\tau,0]$ by $\phi(t) = 1$, and for $t \ge 0$, by

$$\phi(t) = e^{\int_0^t \sigma\tilde{h}(X(s))\,dB(s) - \frac{1}{2}\sigma^2\int_0^t \tilde{h}(X(s))^2\,ds}. \tag{29}$$

Observe that $\phi(t) > 0$ for all $t \ge 0$, a.s., as

$$\frac{1}{2}\sigma^2\int_0^t \tilde{h}(X(s))^2\,ds \;\le \frac{1}{2}\sigma^2\overline{h}^2 t$$

for all $t \ge 0$, and $\int_0^t \tilde{h}(X(s))\,ds$ is well-defined for $0 \le t < \infty$, a.s., and cannot reach $\pm\infty$ in finite time, a.s. To see this, suppose to the contrary that there exists an a.s. finite random variable $T > 0$ such that

$$\lim_{t \uparrow T} \int_0^t \tilde{h}(X(s))\, dB(s) = -\infty, \quad \text{with positive probability.} \tag{30}$$

Letting $M(t) = \int_0^t \tilde{h}(X(s))\, dB(s)$, the martingale time change theorem (cf. e.g. Karatzas and Shreve [6] p.174-5) assures the existence of a standard Brownian motion $\tilde{B}$ such that $M(t) = \tilde{B}(\langle M\rangle(t))$, so (30) implies $\lim_{t\uparrow\langle M\rangle(T)} \tilde{B}(t) = -\infty$, with positive probability. As $\langle M\rangle(T)$ is a.s. finite, we get the desired contradiction. Note further that (29) gives

$$d\phi(t) = \sigma\tilde{h}(X(t))\phi(t)\, dB(t), \tag{31}$$

Define the process Z by $Z(t) = X(t)\phi(t)^{-1}$, $t \geq -\tau$. Then $Z(t) = \psi(t)$ for $t \in [-\tau, 0]$. As $\phi(t) > 0$ for all $t \geq 0$, a.s., we can use (31) and Itô's rule to obtain a semimartingale decomposition of $\phi(t)^{-1}$. Then, by (26) and the definition of Z, integration by parts yields

$$Z(t) = 1 + \int_0^t \phi(s)^{-1} f(s, X_s)\, ds, \quad t \geq 0.$$

Since the functions $t \mapsto \phi(t)^{-1}$, $t \mapsto f(t, X_t)$ are continuous, Z is continuously differentiable on $\mathbb{R}^+$, and thus using (25) we obtain

$$|Z'(t)| \leq K\phi(t)^{-1} \sup_{t-\tau\leq s\leq t} \phi(s)|Z(s)| \leq g(t) \sup_{t-\tau\leq s\leq t} |Z(s)|,$$

where $g(t) = K\phi(t)^{-1}\sup_{t-\tau\leq s\leq t}\phi(s)$ for $t \geq 0$. Let $Z^*(t) = \sup_{t-\tau\leq s\leq t}|Z(s)|$ for $t \geq \tau$. Then, for $t \geq \tau$ we get $Z^*(t) \leq |Z(\tau)| + \int_\tau^t g(s)Z^*(s)\, ds$, so Gronwall's inequality implies $|Z(t)| \leq Z^*(t) \leq |Z(\tau)|\exp(\int_\tau^t g(s)\, ds)$, for $t \geq \tau$. Assume temporarily that

$$\limsup_{t\to\infty} \frac{1}{t}\int_\tau^t \phi(s)^{-1} \sup_{s-\tau\leq u\leq s}\phi(u)\, ds \leq 2\frac{\overline{h}^2}{\underline{h}^2} e^{\sigma^2\overline{h}^2\tau}, \quad \text{a.s.} \tag{32}$$

Then (32) implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log|Z(t)| \leq 2K\frac{\overline{h}^2}{\underline{h}^2}e^{\sigma^2\overline{h}^2\tau}, \quad \text{a.s.} \tag{33}$$

We now obtain an estimate on the top Liapunov exponent of ϕ. Let $M(t) = \int_0^t -\sigma\tilde{h}(X(s))\, dB(s)$. Then (24) yields $\langle M\rangle(t) \leq \sigma^2\overline{h}^2 t$, and so the law of large numbers implies $M(t)/t \to 0$ as $t \to \infty$, a.s. As $|\tilde{h}(x)| \geq \underline{h}$, we get $\limsup_{t\to\infty}\log\phi(t)/t \leq -\sigma^2\underline{h}^2/2$, a.s. This bound, together with the definition of Z and (33) establishes (27), so proving (32) now suffices. The continuity of ϕ ensures that for each $t \geq 0$ there exists a bounded random function μ with $0 \leq \mu(t) \leq \tau$, for $t \geq 0$, a.s. so that

$$\phi(t)^{-1}\sup_{t-\tau\leq s\leq t}\phi(s) = \phi(t)^{-1}\phi(t-\mu(t)).$$

Hence, with

$$P(t) = \int_\tau^t e^{-\int_{s-\mu(s)}^s \sigma\tilde{h}(X(u))\,dB(u)}\,ds, \quad t \geq \tau$$

we have $\int_\tau^t \phi(s)^{-1} \sup_{s-\tau\leq u\leq s} \phi(u)\,ds \leq e^{\frac{1}{2}\sigma^2\overline{h}^2\tau} P(t)$, for $t \geq \tau$. Note that $\langle M\rangle(t) \geq \sigma^2\underline{h}^2 t$, so $\langle M\rangle(t) \to \infty$ as $t\to\infty$, a.s. Therefore there exists a standard Brownian motion $\tilde{B}$ such that $M(t) = \tilde{B}(\langle M\rangle(t))$, $t \geq 0$ a.s. Indeed, $t \mapsto \langle M\rangle(t)$ is continuously differentiable and strictly increasing, with $\langle M\rangle'(t) = \sigma^2\tilde{h}(X(t))^2$. Setting $s(w) = \langle M\rangle^{-1}(w)$, $w \geq \langle M\rangle(\tau)$, we can define

$$\tau(w) = \int_{s(w)-\mu(s(w))}^{s(w)} \sigma^2\tilde{h}(X(u))^2\,du (= \langle M\rangle(s) - \langle M\rangle(s-\mu(s))). \tag{34}$$

Hence, we get

$$\begin{aligned} P(t)\sigma^2\underline{h}^2 &\leq \int_\tau^t e^{-\int_{s-\mu(s)}^s \sigma\tilde{h}(X(u))\,dB(u)} \langle M\rangle'(s)\,ds \\ &= \int_\tau^t e^{M(s)-M(s-\mu(s))}\langle M\rangle'(s)\,ds = \int_{\langle M\rangle(\tau)}^{\langle M\rangle(t)} e^{\tilde{B}(w)-\tilde{B}(w-\tau(w))}\,dw. \end{aligned} \tag{35}$$

Clearly (34) gives $0 \leq \tau(w) \leq \sigma^2\overline{h}^2\tau$ for all $w \geq \langle M\rangle(\tau)$, a.s. Now we may apply Lemma 8 with $\overline{\tau} = \sigma^2\overline{h}^2\tau$, so that, by (35)

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\int_\tau^t \phi(s)^{-1} \sup_{s-\tau\leq u\leq s}\phi(u)\,du \leq \frac{e^{\frac{1}{2}\sigma^2\overline{h}^2\tau}}{\sigma^2\underline{h}^2} \limsup_{t\to\infty} \frac{P(t)\sigma^2\underline{h}^2}{\langle M\rangle(t)}\cdot\frac{\langle M\rangle(t)}{t} \\ &\leq \frac{e^{\frac{1}{2}\sigma^2\overline{h}^2\tau}}{\sigma^2\underline{h}^2}\cdot\sigma^2\overline{h}^2 \limsup_{t\to\infty}\frac{1}{\langle M\rangle(t)}\int_{\langle M\rangle(\tau)}^{\langle M\rangle(t)} e^{\tilde{B}(w)-\tilde{B}(w-\tau(w))}\,dw \leq 2e^{\sigma^2\overline{h}^2\tau}\frac{\overline{h}^2}{\underline{h}^2}, \quad \text{a.s.} \end{aligned}$$

This secures the required estimate.

6 Conclusion

As a conclusion, this paper shows that delay differential equations and functional differential equations can be stabilised with noise. Although extending knowledge of stochastic stabilisation to this type of evolution, the results here are less general than that in [1]. First, we can be certain only that multiplicative noise will aid in stabilising a delay system, whereas many forms suffice in the non-delay case. Moreover, we have been unable to date to prove a result showing that a delay equation with a linear diffusion term can be unstable, even when the corresponding deterministic equation is stable. Such a result seems plausible in view of the close correspondence between the stabilisation mechanism in the delay and non-delay problems. We hope in later work to extend the methodology of this paper to prove such a destabilisation result.

7 Proof of auxiliary results

PROOF (Lemma 8) For each fixed $t \geq \overline{\tau}$, as $0 \leq \tau(t) \leq \overline{\tau}$, almost surely, we have $B(t) - B(t-\tau(t)) \leq \max_{t-\overline{\tau}\leq s\leq t} B(t) - B(s) = \max_{0\leq s\leq \overline{\tau}} W_s(t)$, where $s \mapsto W_s(t)$ is a standard Brownian motion. As $\max_{0\leq s\leq\overline{\tau}} W_s(t)$ has the same distribution as $|W_{\overline{\tau}}(t)|$, and $\mathbb{E}[e^{\lambda|X|}] \leq 2e^{\frac{1}{2}\lambda^2\alpha^2}$ for $X \sim \mathcal{N}(0, \alpha^2)$, we get

$$\mathbb{E}[e^{B(t)-B(t-\tau(t))}] \leq \mathbb{E}[e^{\max_{0\leq s\leq\overline{\tau}} W_s(t)}] = \mathbb{E}[e^{|W_{\overline{\tau}}(t)|}] \leq 2e^{\frac{1}{2}\overline{\tau}}, \quad \text{and} \tag{36}$$

$$\mathbb{E}[e^{4(B(t)-B(t-\tau(t)))}] \leq 2e^{8\overline{\tau}}. \tag{37}$$

Define for $n \in \mathbb{N}$ the two sequences of random variables

$$X_n = \int_{(2n+1)\overline{\tau}}^{(2n+2)\overline{\tau}} e^{B(s)-B(s-\tau(s))}\,ds, \quad Y_n = \int_{2n\overline{\tau}}^{(2n+1)\overline{\tau}} e^{B(s)-B(s-\tau(s))}\,ds. \tag{38}$$

Hence (36) implies that $\mathbb{E}[X_n] \leq 2\overline{\tau}e^{\frac{1}{2}\overline{\tau}}$, $\mathbb{E}[Y_n] \leq 2\overline{\tau}e^{\frac{1}{2}\overline{\tau}}$. By Hölder's inequality and (37), we also get $\mathbb{E}[X_n^4] \leq 2\overline{\tau}^4 e^{8\overline{\tau}}$, $\mathbb{E}[Y_n^4] \leq 2\overline{\tau}^4 e^{8\overline{\tau}}$. Finally, as $0 \leq \tau(t) \leq \overline{\tau}$, we see that (X_n) is a sequence of independent random variables, as is (Y_n). Note now that a version of the Strong Law of Large Numbers (see, for instance [7], p.72-3) implies

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^n X_j \leq 2\overline{\tau}e^{\frac{1}{2}\overline{\tau}}, \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^n Y_j \leq 2\overline{\tau}e^{\frac{1}{2}\overline{\tau}}, \quad \text{a.s.}$$

With $t \geq 4\overline{\tau}$ there is $n = n(t) \in \mathbb{N}$ such that $(2n+2)\overline{\tau} \leq t < (2n+4)\overline{\tau}$ and

$$\frac{1}{t}\int_{2\overline{\tau}}^t e^{B(s)-B(s-\tau(s))}ds = \frac{n}{t}\left(\frac{1}{n}\sum_{j=1}^n X_j + \frac{1}{n}\sum_{j=1}^n Y_j\right) + \frac{1}{t}\int_{(2n+2)\overline{\tau}}^t e^{B(s)-B(s-\tau(s))}ds.$$

Since $\lim_{t\to\infty} n(t)/t = \frac{1}{2\overline{\tau}}$, if we can show

$$\lim_{t\to\infty} \frac{1}{t}\int_{(2n(t)+2)\overline{\tau}}^t e^{B(s)-B(s-\tau(s))}\,ds = 0, \quad \text{a.s.} \tag{39}$$

(28) is true. Notice now that (39) is true if $\lim_{n\to\infty} U_n = 0$ a.s. where $U_n = \frac{X_n+Y_n}{n+1}$. But this is true by Markov's inequality and the Borel-Cantelli lemma as $\mathbb{E}[U_n^4] \leq C(n+1)^{-4}$, for some $C > 0$.

PROOF (Lemma 1) As $\varphi(s)^{-1}\varphi(s-\tau(s)) \leq e^{\sigma^2\overline{\tau}/2}e^{-\sigma(B(s)-B(s-\tau(s)))}$, we need

$$\limsup_{t\to\infty} \frac{1}{t}\int_{2\overline{\tau}}^t e^{-\sigma(B(s)-B(s-\tau(s)))}\,ds \leq e^{\sigma^2\overline{\tau}/2}. \tag{40}$$

Next, set $a_0 = 2\overline{\tau}$, and define the increasing sequence a_n by $\int_{a_n}^{a_{n+1}} e^{\frac{1}{2}\sigma^2\tau(s)}\,ds = \overline{\tau}e^{\frac{1}{2}\overline{\tau}}$. Since $0 \le \tau(t) \le \overline{\tau}$, we get $\overline{\tau} \le a_{n+1} - a_n \le \overline{\tau}e^{\frac{1}{2}\sigma^2\overline{\tau}}$. Define

$$X_n = \int_{a_{2n+1}}^{a_{2n+2}} e^{-\sigma(B(s)-B(s-\tau(s)))}\,ds, \quad Y_n = \int_{a_{2n}}^{a_{2n+1}} e^{-\sigma(B(s)-B(s-\tau(s)))}\,ds,$$

for $n = 0, 1, \ldots$. By the construction of the sequence $(a_n)_{n\ge0}$, the sequence of random variables $(X_n)_{n\ge0}$ are independently distributed, as are $(Y_n)_{n\ge0}$. Note moreover that $\mathbb{E}[X_n] = \mathbb{E}[Y_n] = \overline{\tau}e^{\frac{1}{2}\sigma^2\overline{\tau}}$. With $\mathbb{E}[X_n^4] \le K$, $\mathbb{E}[Y_n^4] \le K$ for some $K > 0$, the Strong Law of Large numbers implies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} X_j = \overline{\tau}e^{\frac{1}{2}\sigma^2\overline{\tau}}, \quad \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} Y_j = \overline{\tau}e^{\frac{1}{2}\sigma^2\overline{\tau}}, \quad \text{a.s.}$$

The proof concludes as that of Lemma 8, and is therefore omitted. □

PROOF (Lemma 2) Set $q(t) = \int_{t-\overline{\tau}}^{t} e^{-\sigma(B(t)-B(s))}\,ds$. Then q_1 in Lemma 2 satisfies $q_1(t) \le e^{\frac{\sigma^2}{2}\overline{\tau}}q(t)$, and proving

$$\lim_{t\to\infty} \frac{1}{t}\int_{2\overline{\tau}}^{t} q(s)\,ds = \int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u}\,du, \quad \text{a.s.,} \tag{41}$$

suffices. Then $X_n = \int_{(2n+1)\overline{\tau}}^{(2n+2)\overline{\tau}} q(s)\,ds$, $Y_n = \int_{2n\overline{\tau}}^{(2n+1)\overline{\tau}} q(s)\,ds$, $n \ge 1$, are sequences of independent random variables, with $\mathbb{E}[X_n] = \mathbb{E}[Y_n] = \overline{\tau}\int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u}\,du$. The proof of (41) relies on the same method as used in Lemma 1. □

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