Non-exponential Stability of Scalar Stochastic Volterra Equations

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Abstract

We study convergence rates to zero of solutions of the scalar equation

$$dX(t) = \left(f(X(t)) + \int_0^t k(t-s)g(X(s))\,ds \right) dt + h(X(t))\,dB(t),$$

where f, g, h are globally Lipschitz, $xg(x) > 0$ for non-zero x, and k is continuous, integrable, positive and $\lim_{t\to\infty} k(t-s)/k(t) = 1$, for $s > 0$. Then

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty \quad \text{a.e. on } A,$$

for nontrivial solutions satisfying $\lim_{t\to\infty} X(t) = 0$ on A, a set of positive probability.

Key words: almost sure exponential asymptotic stability, Itô-Volterra equations, stochastic integrodifferential equations
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1 Introduction

The theory of deterministic nonautonomous time–homogeneous linear functional differential equations with bounded delay, contains results showing the

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equivalence of exponential asymptotic stability and uniform asymptotic stability for equilibria. However for Volterra equations with unbounded delay this equivalence need not hold. For instance, [1] considered the scalar problem

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,ds, \tag{1}$$

with the kernel k continuous, integrable and of a single sign. He showed that the uniform asymptotic stability and the exponential asymptotic stability of the zero solution of this equation are equivalent if and only if

$$\int_0^\infty |k(s)|e^{\gamma s}\,ds < \infty \tag{2}$$

for some $\gamma > 0$. This condition is also referred to as exponential integrability. Hence if it fails to hold, a uniformly asymptotically stable solution of (1) cannot be exponentially asymptotically stable. This poses the question of what is the nonexponential decay rate to zero.

[2] investigated this problem for kernels satisfying

$$\lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0, \tag{3}$$

a condition which implies $\lim_{t\to\infty} k(t)e^{\gamma t} = \infty$ for all $\gamma > 0$ and therefore prevents k from being exponentially integrable. It is shown there that if the zero solution is asymptotically stable, then

$$\liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \geq \frac{|x(0)|}{a\Big(a - \int_0^\infty k(s)\,ds\Big)}. \tag{4}$$

In this paper, we investigate a more general stochastic version of (1), namely the scalar nonlinear Itô-Volterra equation

$$dX(t) = \left(f(X(t)) + \int_0^t k(t-s)g(X(s))\,ds\right)dt + h(X(t))\,dB(t). \tag{5}$$

As a special case, we also consider the stochastic version of (1):

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma X(t)\,dB(t). \tag{6}$$

The method employed in [2] is extended here to investigate these problems under a weaker hypothesis on the kernel k than (3), namely

$$\lim_{t\to\infty} \frac{k(t-s)}{k(t)} = 1 \quad \text{uniformly for } s \text{ on compact intervals.}$$

It is shown here that if either of the equations (6) or (5) have nontrivial strong solutions $X(t)$ which tend to zero on a set of positive probability A, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty \quad \text{a.s. on } A.$$

This precludes the exponential asymptotic stability of the zero solution on any set of positive probability. It also asserts the convergence to zero of $X(t)$ as $t \to \infty$ is no faster than that of $k(t)$.

There is a significant literature devoted to the almost sure exponential asymptotic stability of equilibrium solutions of stochastic differential equations and stochastic differential equations with bounded delay. In particular we highlight the papers of [3], [4], [5], [6] and [7], and the monographs [8] and [9]. Nonexponential almost sure asymptotic stability has also been studied in [10], [11] and [12], but the nonexponential convergence arises from the non-autonomous structure of the equations considered. The slower than exponential convergence which can arise in certain nonlinear autonomous stochastic differential equations is considered for example in [13]. Results relating to the asymptotic stability of solutions of Itô-Volterra equations can be found in [14] and [15].

2 Main Results

In this paper, $(B(t))_{t\geq 0}$ is a standard one-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{\geq 0}, \mathbb{P})$, where the filtration is the natural one, viz., $\mathcal{F}^B(t) = \sigma(B(s) : 0 \leq s \leq t)$. When almost sure events are referred in this paper, they are always $\mathbb{P}$-almost sure. Consider the general nonlinear scalar (stochastic) Itô-Volterra equation

$$dX(t) = \left(f(X(t)) + \int_0^t k(t-s) g(X(s))\, ds \right) dt + h(X(t))\, dB(t), \tag{7}$$

The kernel satisfies

$$k(t) \geq 0, \quad k \in C[0,\infty). \tag{8}$$

To ensure that k is not exponentially integrable, the following additional condition was imposed in [2]:

$$k \in C^1[0,\infty), \quad k(t) > 0 \text{ for } t \geq 0, \quad \lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0. \tag{9}$$

Consequently, for every $\gamma > 0$,

$$\lim_{t\to\infty} k(t) e^{\gamma t} = \infty. \tag{10}$$

Here, we prefer the weaker hypothesis on k:

$$k(t) > 0 \text{ for } t \geq 0, \quad \lim_{t\to\infty} \frac{k(t-s)}{k(t)} = 1 \quad \text{uniformly for } s \in [0,T]. \tag{11}$$

Note that the third condition of (9) implies the second in (11), as was pointed out in [2], and (10) can be inferred from (11).

We now make some hypotheses concerning the functions f, g, h. It is assumed that:

$$f,\, g,\, h \text{ are globally Lipschitz continuous}, \quad f(0) = g(0) = h(0) = 0. \tag{12}$$

Thus they satisfy global linear bounds of the form $|f(x)| \vee |g(x)| \vee |h(x)| \leq L|x|$ for all $x \in \mathbb{R}$ for some $L > 0$. These conditions, (8) imply that (7) has a unique continuous strong solution, once an initial value $X(0)$ has been supplied which is independent of the Brownian motion B, and which obeys $\mathbb{E}[X(0)^2] < \infty$ (cf. [16, Theorem 2E]). Moreover, if $X(0) = 0$, then $X(t) = 0$ for all $t \geq 0$ almost surely. This solution is called the zero solution of (7).

We now state the main results of the paper, and comment upon them.

Theorem 1 *Let k satisfy (8) and (11), and f, g, h obey* (12). *Suppose that f, h are continuously differentiable at the origin with $h'(0) \neq 0$, and $xg(x) > 0$ for $x \neq 0$. Let X be a nontrivial strong solution of*

$$dX(t) = \left(f(X(t)) + \int_0^t k(t-s) g(X(s))\, ds \right) dt + h(X(t))\, dB(t) \tag{13}$$

which obeys $\lim_{t\to\infty} X(t) = 0$ on a set A of positive probability. Then

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty, \quad \textit{a.e. on } A, \tag{14}$$

and for every $\varepsilon > 0$,

$$\limsup_{t\to\infty} |X(t)| e^{\varepsilon t} = \infty \quad \textit{a.e. on } A. \tag{15}$$

Although the hypothesis $k \in L^1(0,\infty)$ is not used in the proof of Theorem 1, it is usually implicit. Indeed, motivated by the fact that the integrability of k is required to prove the asymptotic stability in the deterministic case, we may prove that the requirements $X(t) \to 0$ as $t \to \infty$ and $X \in L^1(0,\infty)$ on a set of positive probability imply $k \in L^1(0,\infty)$. In the other case when $X(t) \to \infty$ as $t \to \infty$, and $X \notin L^1(0,\infty)$, the conclusion (15) is immediately apparent.

As a corollary of Theorem 1, we have the following result for the linear equation

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma X(t)\,dB(t). \tag{16}$$

As (16) is linear, we may assume $X(0) = 1$ without loss of generality.

Corollary 2 *Let k satisfy (8) and (11), and $\sigma \neq 0$. Let X be a nontrivial strong solution of*

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma X(t)\,dB(t), \quad X(0) = 1, \tag{17}$$

which obeys $\lim_{t\to\infty} X(t) = 0$ on a set A of positive probability. Then

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty \quad \text{a.e. on } A, \tag{18}$$

and for every $\varepsilon > 0$,

$$\limsup_{t\to\infty} |X(t)|e^{\varepsilon t} = \infty \quad \text{a.e. on } A. \tag{19}$$

It is easy to deduce from this theorem a result for strong solutions of (16) with general nontrivial initial condition, owing to the linearity of (16).

We note that (19), which is a trivial consequence of (10) and (18), demonstrates that $X(t)$ cannot converge exponentially fast to 0 as $t \to \infty$. Corollary 2 hypothesises the existence of a strong solution converging almost surely to zero. Elsewhere, we show that a sufficient condition to guarantee the a.s. convergence of solutions of (16) and (17) to zero is $a > \int_0^\infty k(s)\,ds$.

By virtue of Remark 4 below, we could replace the strong hypothesis that $X(t) \to 0$ as $t \to \infty$ with the weaker parameter restriction $a + \sigma^2/2 > 0$. However, we favour the hypothesis of Corollary 2 as stated, as it serves to emphasise the central point of our note: that solutions of linear Itô-Volterra equations need not be exponentially asymptotically stable on a set of positive probability, even when they are asymptotically stable on that set. Moreover, we retain this asymptotic stability hypothesis as there is not, to our knowledge, a set of conditions on the parameters a, σ, and kernel k yet present in the literature which characterises, for example, the almost sure asymptotic stability of solutions of (16).

The results in [17] together with Corollary 2 show that linear noise perturbations can slow the decay rates of asymptotically stable solutions of scalar Volterra equations. This is in contrast to the stabilising effect of a noise perturbation of the same form for the linear stochastic differential equation

$$dX(t) = -aX(t)\,dt + \sigma X(t)\,dB(t).$$

Indeed, for the class of subexponential kernels introduced in [17], $x(t)/k(t)$ tends to a nonzero limit as $t \to \infty$. Therefore Corollary 2 shows that the rate of convergence of $X(t)$ to zero is slower.

3 Proof of Theorem 1

To prove Theorem 1, the following technical result is first required.

Lemma 3 *For every $\tau > 0$, $\sigma > 0$, if B is a standard Brownian motion, then*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} e^{\sigma(B(t)-B(s))}\,ds = \infty \quad a.s. \tag{20}$$

PROOF. Observe that the sequence of random variables

$$U_n = \int_{(n-1)\tau}^{n\tau} e^{\sigma(B(n\tau)-B(s))}\,ds, \quad n = 1, 2, \ldots.$$

are independently and identically distributed, each with the same distribution as the strictly positive random variable $U = \int_0^\tau e^{\sigma(B(\tau)-B(s))}\,ds$. Therefore, if we can show that

$$\mathbb{P}[U > C] > 0 \quad \text{for all } C > 0, \tag{21}$$

the second Borel-Cantelli Lemma implies that $\limsup_{n\to\infty} U_n = \infty$ a.s., and so (20) follows. To prove (21), suppose to the contrary that there is $C > 0$ such that $\mathbb{P}[U \le C] = 1$. For $m \ge 0$, define

$$A_m = \{\omega \in \Omega : B(\tau)(\omega) - \max_{0\le s\le \tau/2} B(s)(\omega) > m/\sigma\}$$

The random variable $B(\tau) - \max_{0\le s\le\tau/2} B(s)$ has the same distribution as $W^{(1)}(\tau/2) + \min_{0\le s\le\tau/2} W^{(2)}(s)$, where $W^{(1)}$, $W^{(2)}$ are independent standard Brownian motions. Thus $\mathbb{P}[A_m] > 0$ for all $m \ge 0$. Define $m = \log(4C/\tau) \vee 0$. Then, for $\omega \in A_m$

$$U(\omega) \ge \int_{\tau/2}^{\tau} e^{\sigma(B(\tau)-B(s))}\,ds \ge \int_{\tau/2}^{\tau} e^{m}\,ds \ge 2C > C.$$

Thus $\mathbb{P}[U > C] \ge \mathbb{P}[A_m] > 0$, a contradiction. □

We now turn to the proof of Theorem 1.

PROOF (Theorem 1) Observe that equation (13) can be rewritten as

$$dX(t) = \left(\tilde{f}(X(t))X(t) + \int_0^t k(t-s)g(X(s))\,ds\right)\,dt \\ + \sigma X(t)\tilde{h}(X(t))\,dB(t), \tag{22}$$

where $\sigma = h'(0) \neq 0$, and $\tilde{f}$, $\tilde{h}$ are the bounded, continuous functions defined by

$$\tilde{f}(x) = \begin{cases} f(x)/x, \; x \neq 0, \\ f'(0), \quad x = 0, \end{cases} ; \quad \tilde{h}(x) = \begin{cases} 1/\sigma \cdot h(x)/x, \; x \neq 0, \\ 1, \qquad\qquad x = 0. \end{cases}$$

Without loss of generality $\sigma > 0$. As observed previously the global Lipschitz conditions on f, h imply that $|f(x)| \vee |h(x)| \leq L|x|$ for all $x \in \mathbb{R}$, and some $L > 0$. We assume without loss that $X(0) > 0$.

Define the process ϕ by

$$\frac{\phi(t)}{X(0)} = \exp\left(\int_0^t \left[\tilde{f}(X(s)) - \frac{\sigma^2\tilde{h}^2(X(s))}{2}\right] ds + \int_0^t \sigma\tilde{h}(X(s))\,dB(s)\right). \tag{23}$$

Notice that $\phi(t) > 0$ for all $t \geq 0$ a.s., and note that ϕ obeys

$$d\phi(t) = \tilde{f}(X(t))\phi(t)\,dt + \sigma\tilde{h}(X(t))\phi(t)\,dB(t). \tag{24}$$

As $\phi^{-1}(t)$ is well-defined for all $t \geq 0$ a.s., we may define

$$Z(t) = X(t)\phi^{-1}(t), \quad t \geq 0. \tag{25}$$

Hence, $Z(0) = 1$, so using (22), (24) and integration by parts we get

$$Z(t) = 1 + \int_0^t \phi(s)^{-1} \int_0^s k(s-u)g(\phi(u)Z(u))\,du\,ds. \tag{26}$$

To prove that $Z(t) \geq 1$ for all $t \geq 0$, and hence that $X(t) \geq \phi(t) > 0$ for all $t \geq 0$, a.s., by (25) and the positivity of k, ϕ and g, it is enough to prove that

$$Z(t) > 0 \quad \text{for all } t \geq 0 \text{ a.s.,} \tag{27}$$

on account of (26). We denote by Ω^* the almost sure subset of Ω on which (26) holds and $\phi(t) > 0$ for all $t \geq 0$.

Consider the set $\Omega_0 = \{\omega \in \Omega^* : Z(t)(\omega) = 0 \text{ for some finite } t > 0\}$. For each $\omega \in \Omega_0$, define $T_\omega = \inf\{t \geq 0 : Z(t)(\omega) = 0\}$. Since $Z(0) = 1$ and $t \mapsto Z(t)(\omega)$ is continuous, $T_\omega > 0$. As $g(x) > 0$ for all $x > 0$, $g(Z(t)\phi(t))(\omega)$ is positive on $[0, T_\omega)$. Since k is positive on $[0, T_\omega)$, it follows from (26) that $0 = Z(T_\omega)(\omega) \geq 1$. This contradiction shows that Ω_0 is empty, and (27) follows.

We now prove that

$$\liminf_{t\to\infty} \int_0^t \frac{k(t-s)g(X(s))}{k(t)}\,ds \geq \int_0^\infty g(X(s))\,ds, \quad \text{a.s.} \tag{28}$$

This holds true even when $g \circ X \notin L^1[0,\infty)$ and the righthand side of the inequality is interpreted as infinity.

For all $t > T$, we get

$$\int_0^t \frac{k(t-s)g(X(s))}{k(t)}\,ds \geq \int_0^T \left(\frac{k(t-s)}{k(t)} - 1\right) g(X(s))\,ds + \int_0^T g(X(s))\,ds.$$

Since

$$\left|\int_0^T \left(\frac{k(t-s)}{k(t)} - 1\right) g(X(s))\,ds\right| \leq \max_{0\leq s\leq T} \left|\frac{k(t-s)}{k(t)} - 1\right| \int_0^T g(X(s))\,ds,$$

for $t > T$, we get

$$\int_0^t \frac{k(t-s)g(X(s))}{k(t)}\,ds \geq \left(1 - \max_{0\leq s\leq T}\left|\frac{k(t-s)}{k(t)} - 1\right|\right) \int_0^T g(X(s))\,ds.$$

Hence equation (11) implies that

$$\liminf_{t\to\infty} \int_0^t \frac{k(t-s)g(X(s))}{k(t)}\,ds \geq \int_0^T g(X(s))\,ds, \quad \text{a.s.}$$

Letting $T \to \infty$ yields (28).

Next, by (28), it follows that there exists $T_1(\omega)$, $L_1(\omega)$ both finite and positive such that

$$\int_0^t \frac{k(t-s)g(X(s))}{k(t)}\,ds \geq L_1, \quad t \geq T_1.$$

Hence, from (25), (26), the positivity of ϕ, k, and the above estimate, we get

$$\frac{X(t)}{k(t)} \geq L_1 \frac{\phi(t)}{k(t)} \int_{T_1}^t k(s)\phi(s)^{-1}\,ds, \quad t \geq T_1. \tag{29}$$

For $t \geq T_1$, define

$$Y(t) = \frac{\phi(t)}{k(t)} \int_{T_1}^t k(s)\phi(s)^{-1}\,ds. \tag{30}$$

Next, let $\tau > 0$ be deterministic. By (11) there exists $t_2 > 2\tau$ such that for all $t \geq t_2$, $k(t-s) > k(t)/2$ whenever $s \in [0, 2\tau]$. Let $T_2 = t_2 \vee (T_1 + 2\tau)$. Then for $t \geq T_2$ we therefore obtain

$$Y(t) \geq \frac{1}{2}\phi(t) \int_{t-2\tau}^t \phi(s)^{-1}\,ds.$$

Hence

$$Y(t) \geq \frac{1}{2} \int_{t-2\tau}^{t} e^{\int_s^t \tilde{f}(X(u)) - \frac{1}{2}\sigma^2 \tilde{h}^2(X(u))\, du + \int_s^t \sigma \tilde{h}(X(u))\, dB(u)}\, ds,$$

so the boundedness of $\tilde{f}$, $\tilde{h}$ ensure the existence of a deterministic $C_1 > 0$ such that

$$Y(t) \geq C_1 \int_{t-2\tau}^{t} e^{\sigma \int_s^t \tilde{h}(X(u))\, dB(u)}\, ds, \quad t \geq T_2.$$

From this point on in the proof, our analysis is confined to the set of positive probability $A = \{\omega \in \Omega \,:\, X(t)(\omega) \to 0 \text{ as } t \to \infty\}$. In order to avoid writing the ω-dependence of random variables explicitly on every occasion, we make this qualification explicit only where necessary.

As $\tilde{h}(X(t)) \to 1$ as $t \to \infty$ on A, we have $1/2 < \tilde{h}^2(X(t)) < 2$ for all $t > T_2'$. Thus, for $T_3 = (T_2' + 2\tau) \vee T_2$, and $C_2 = C_1/(2\sigma^2)$, for $t > T_3$ we have

$$Y(t) \geq C_2 \int_{t-2\tau}^{t} \sigma^2 \tilde{h}^2(X(s)) e^{\sigma \int_s^t \tilde{h}(X(u))\, dB(u)}\, ds. \tag{31}$$

Now, define $M(t) = \int_0^t \sigma \tilde{h}(X(s))\, dB(s)$. Clearly, by the construction of T_3, we have $\sigma \tilde{h}(X(t))(\omega) > 0$ for all $t > T_3$ and $\omega \in A$. Hence, $t \mapsto \langle M \rangle(t)$ is increasing on (T_3, ∞). Indeed, when restricted to A, as $\tilde{h}(X(t)) \to 1$ as $t \to \infty$, $\langle M \rangle(t) \to \infty$ as $t \to \infty$. Hence, by the martingale time change theorem (cf. e.g, [18, p.174-5]) there exists a standard Brownian motion $\tilde{B}$ such that $M(t) = \tilde{B}(\langle M \rangle(t))$, for all $t \geq 0$, almost everywhere on A.

Consider now $T > T_2' + 4\tau/\sigma^2$. Then

$$\int_{T-4\tau/\sigma^2}^{T} \sigma^2 \tilde{h}(X(s))^2\, ds > 2\tau.$$

Therefore, there is a well-defined function $T^* : [T_2' + 4\tau/\sigma^2, \infty) \to [T_2', \infty) : T \mapsto T^*(T)$ such that

$$\int_{T^*(T)}^{T} \sigma^2 \tilde{h}^2(X(s))\, ds = 2\tau,$$

a.e. on A. Observe that $T^*(T) \to \infty$ as $T \to \infty$. For $T > T_2' + 4\tau/\sigma^2$, we have $T^*(T) > T_2'$ and hence

$$2\tau = \int_{T^*(T)}^{T} \sigma^2 \tilde{h}^2(X(s))\, ds \leq (T - T^*(T)) 2\sigma^2.$$

This implies $T - T^*(T) \geq \tau'$, where $\tau' = \tau/\sigma^2$. As $t \mapsto \langle M \rangle(t)$ is increasing on (T_3, ∞), so $t^* = \langle M \rangle^{-1}(T_3 \vee (T_2' + 4\tau/\sigma^2))$ is well-defined. Now, suppose $t > t^*$. Then we can always define $T > T_3 \vee (T_2' + 4\tau/\sigma^2)$ so that $T = \langle M \rangle(t)$ and

$$\int_{t-2\tau}^{t} \sigma^2 \tilde{h}^2(X(s)) e^{\sigma \int_s^t \tilde{h}(X(u))\, dB(u)}\, ds = \int_{T^*(T)}^{T} e^{\tilde{B}(T) - \tilde{B}(w)}\, dw. \tag{32}$$

As $T - T^*(T) \geq \tau'$ for $T > T_3 \vee (T_2' + 4\tau/\sigma^2)$, we have

$$\int_{T^*(T)}^{T} e^{\tilde{B}(T)-\tilde{B}(w)}\,dw \geq \int_{T-\tau'}^{T} e^{\tilde{B}(T)-\tilde{B}(w)}\,dw. \tag{33}$$

As $t \to \infty$, $T = \langle M\rangle(t) \to \infty$, so by (31), (32), (33) we can prove that $\limsup_{t\to\infty} Y(t) = \infty$ a.e. on A provided

$$\limsup_{T\to\infty} \int_{T-\tau'}^{T} e^{\tilde{B}(T)-\tilde{B}(w)}\,dw = \infty, \quad \text{a.e. on } A, \tag{34}$$

which is true by Lemma 3, as (34) holds for a standard Brownian motion on an almost sure set. Returning to (29), (30), we see that (34) implies (14), as required. □

Remark 4

We conclude with a remark alluded to earlier, namely that the hypothesis in Corollary 2 that the strong solution decays to zero places a restriction on the parameters a and σ. More precisely, if k satisfies (8), and the unique strong solution of (17) satisfies $\lim_{t\to\infty} X(t) = 0$ on a set of positive probability, then $a + \sigma^2/2 > 0$.

To see this, we note in the linear case, where we may choose $X(0) = 1$ without loss of generality, $(\phi(t))_{t\geq 0}$ is defined by $\phi(0) = 1$ and

$$d\phi(t) = -a\phi(t)\,dt + \sigma\phi(t)\,dB(t).$$

In the proof of Theorem 1 we showed that Z defined by (25) obeys $Z(t) \geq 1$, for all $t \geq 0$, a.s., so $X(t) \geq \phi(t)$. However, for $a+\sigma^2/2 \leq 0$, $\limsup_{t\to\infty} \phi(t) = \infty$, a.s., so for $X(t) \to 0$ as $t \to \infty$ on a set of positive probability, we must have $a + \sigma^2/2 > 0$.

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