STABILISATION OF VOLTERRA EQUATIONS BY NOISE

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ABSTRACT. The paper is concerned with the relationship between the stability of the Itô-Volterra equation

$$dX(t) = \left(-af(X(t)) + \int_0^t k(t-s)g(X(s))\,ds \right)\,dt + \sigma X(t)\,dB(t)$$

and its deterministic counterpart (where $\sigma = 0$). It is assumed that $|k|$ is bounded by a negative exponential with exponent $-\mu$. Under mild continuity and growth hypotheses on f, g, we show that there exists an open set in (μ, a) parameter space in which the deterministic problem is unstable, while for an interval of values of $|\sigma|$, all solutions of the stochastic problem tend to zero exponentially fast, almost surely. In addition, for the scalar linear problem when k is a negative exponential, and the deterministic problem is stable, there is an interval of values of σ for which the stochastic problem is stable.

1. Introduction

This paper aims to contribute to research on the question of the stabilisation or destabilisation of a deterministic dynamical system (differential equation, partial differential equation or functional differential equation) by a noise perturbation, and in particular perturbations which transform the differential equation (or FDE) to one of Itô type. Mao has written an interesting paper [9] devoted to the study of stabilisation and destabilisation of nonlinear finite dimensional differential equations, which has been extended to examine the stabilisation of partial differential equations by Mao, Caraballo and Liu [6]. The asymptotic behaviour of linear functional differential equations with bounded delay has been studied by Mohammed and Scheutzow [11], wherein it is shown that time delays in the diffusion coefficient can destablilise a linear functional differential equation. The stabilisation of nonlinear finite dimensional functional differential equations with (sufficiently small) bounded delay has been covered by the author in [3], and the destabilisation of even-dimensional equations in [1]. In the latter paper, however, the delay can be unbounded, so Volterra equations can be destabilised, as a special case. For scalar, linear, convolution Volterra equations with positive and integrable kernel, Appleby has shown in [2] that the corresponding family of Itô-Volterra equations with a diffusion term of the form σx is almost surely asymptotically stable, provided the deterministic problem is uniformly asymptotically stable, so the addition of noise is not destabilising. To the authors' knowledge the issue of stabilisation of a Volterra equation by noise perturbations of Itô type has not, to date, been studied. The question is of interest in applications in the context of the dynamic stability of viscoelastic members subject to stochastic perturbations.

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In this paper, we first consider a very simple type of nonlinear scalar convolution Volterra equation:

$$x'(t) = -af(x(t)) + \int_0^t \lambda e^{-\mu(t-s)} g(x(s))\, ds, \tag{1.1}$$

and its Itô perturbation

$$dX(t) = \left(-af(X(t)) + \int_0^t \lambda e^{-\mu(t-s)} g(X(s))\, ds\right) dt + \sigma X(t)\, dB(t). \tag{1.2}$$

The presence of an exponential kernel is central to our initial analysis: it means that (1.1) (resp. (1.2)) can be written as a differential equation (resp. SDE) in $\mathbb{R}^2$. There are two important benefits of this alternative representation. First, a finite dimensional stochastic differential equation is easier to analyse than a (presumably) infinite dimensional Volterra equation. Second, it is true that solutions of stochastic functional differential equations can oscillate about their zero equilibrium position in the sense that there is no last zero of $X(t)$, almost surely. Therefore we cannot, in general use Itô's rule to obtain the semimartingale decomposition of $|X(t)|^p$ for $0 < p < 1$ (or $\log|X(t)|$), which is very helpful in obtaining good information on the asymptotic stability of stochastic differential equations. For linear equations, Arnold has shown in [4] (under appropriate Lie algebra conditions on the matrices A and B) that the linear stochastic system

$$dX(t) = AX(t)\, dt + BX(t)\, dW(t)$$

has almost sure top Liapunov exponent which is the limit as $p \downarrow 0$ of the top Liapunov exponent of the p-th mean $\mathbb{E}[\|X(t)\|^p]$. However, the system of stochastic differential equations created from (1.2) are non-zero for all time, a.s., so, with $Z(t) = (X(t), Y(t))$, where

$$Y(t) = \int_0^t e^{-\mu(t-s)} X(s)\, ds$$

we can consider the semimartingale $\|Z(t)\|^p$ for small $p > 0$ and use a nonlinear analogue of Arnold's result above (which is due to Mao [10]), to extract an upper bound on the top Liapunov exponent of Z.

Once results on stabilisation have been obtained for the linear case of (1.2), it is then possible to use a type of comparison principle argument to deal with the more general equation

$$dX(t) = \left(-af(X(t)) + \int_0^t k(t-s) g(X(s))\, ds\right) dt + \sigma X(t)\, dB(t) \tag{1.3}$$

when $|k(t)| \le \lambda e^{-\mu t}$, for all $t \ge 0$, and some positive μ and λ. We also extend these results to nonconvolution equations and general finite dimensional equations.

The paper is organised as follows: the details of the problem to be studied, notation and supporting results are presented in Section 2. An upper bound on the p-th mean and a.s. top Liapunov exponent of the equation (1.2) is obtained in Section 3. The bound can be written in terms of an optimisation problem which is parameterised by the model parameters a, μ, λ and the functions f, g. In section 4, we obtain sufficient conditions on the model parameters to ensure that the a.s. top Liapunov exponent of solutions of (1.2) is negative. In Section 5, these results are formally stated, and we show how the region in (μ, a, λ) parameter space for which global a.s. exponential asymptotic stability can be enlarged under certain hypotheses on the sign of f. Indeed, it is established that and we show that deterministic problems of the form (1.1) which have unstable zero solutions can be stabilised for all initial conditions, almost surely, whenever σ lies in a nonempty a, μ, λ-dependent interval. Results on nondestabilisation for linear problems are also established in Section 5. Section 6 has the same concerns as Section 5, but instead covers the more general problem (1.3). In Section 7 our results are applied to some specific problems. Further generalisations are presented in Section 8.

2. The Scalar Problem

We will use the following standard notation in this paper. Let $\mathbb{R} = (-\infty, \infty)$, and $\mathbb{R}^+ = [0, \infty)$. Denote the minimum of x, y in $\mathbb{R}$ by $x \wedge y$, their maximum by $x \vee y$, and the signum function by $\text{sgn}\,(x) = 1$ for $x > 0$, and $\text{sgn}\,(x) = -1$ for $x \leq 0$. If I, J are open sets in a Banach space, we denote the class of continuous functions taking I onto J by $C(I; J)$. Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis vectors in $\mathbb{R}^2$. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, we denote the standard innerproduct of $\mathbf{x}$ and $\mathbf{y}$ by $\langle \mathbf{x}, \mathbf{y} \rangle$. The standard Euclidean norm of $\mathbf{x} \in \mathbb{R}^2$ is given by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. We denote the space of 2×2 matrices with real entries by $M_{2,2}(\mathbb{R})$, so the space of continuous functions taking $\mathbb{R}^2$ onto $\mathbb{M}_{2,2}(\mathbb{R})$ is given by $C(\mathbb{R}^2; M_{2,2}(\mathbb{R}))$. We take the standard operator norm as norm on $M_{2,2}(\mathbb{R})$; for $A \in M_{2,2}(\mathbb{R})$, the operator norm of A is given by

$$\|A\| = \sup\{\mathbf{x} \in \mathbb{R}^2, \|\mathbf{x}\| = 1 \,:\, \|A\mathbf{x}\|\}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We study the scalar Itô-Volterra equation

$$dX(t) = \left(-af(X(t)) + \int_0^t k(t-s) g(X(s))\, ds \right) dt + \sigma X(t)\, dB(t) \tag{2.1}$$

where $a \in \mathbb{R}$, $\sigma \neq 0$, and f, g, k are continuous functions, whose properties we later specify in more detail. Here, $B = \{B(t), \mathcal{F}_t^B; 0 \leq t < \infty\}$ is a standard one dimensional Brownian motion on the probability space, with natural filtration $(\mathcal{F}_t^B)_{t \geq 0}$. We suppose that (2.1) is an initial value problem, with $X(0) = \xi$, where ξ is a square integrable random variable which is independent of B. In the usual way, the filtration $(\mathcal{F}_t^B)_{t\geq 0}$ can be extended to $(\mathcal{F}_t)_{t \geq 0}$ in such a way that $B = \{B(t), \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion.

In this paper, we assume that $k \in C(\mathbb{R}; \mathbb{R})$ and that $\sup_{t \geq 0} |k(t)| e^{\mu t}$ is finite for some $\mu > 0$. In particular, we first study (2.1) where $k(t)$ is a negative exponential, viz., $k(t) = \lambda e^{-\mu t}$ for $t \geq 0$, so

$$dX(t) = \left(-af(X(t)) + \int_0^t \lambda e^{-\mu(t-s)} g(X(s))\, ds \right) dt + \sigma X(t)\, dB(t), \tag{2.2}$$

where $\lambda \neq 0$ and $\mu > 0$. We remark in advance that all results relating to existence and uniqueness of solutions, and to the zero solution of (2.2) apply equally to the more general problem (2.1), under the hypotheses which we impose below on the functions f, g.

Suppose that f, $g \in C(\mathbb{R}; \mathbb{R})$ and suppose that

$$f(0) = 0, \quad g(0) = 0. \tag{2.3}$$

Let f, g satisfy local Lipschitz conditions, and the following global linear bounds:

$$|f(x)| \leq \overline{f}|x|, \quad |g(x)| \leq \overline{g}|x|, \tag{2.4}$$

where $\overline{f}$, $\overline{g}$ are finite positive constants. We also suppose without loss of generality that $\overline{f}$, $\overline{g}$ are optimal in (2.4) above, so that

$$\sup_{x \in \mathbb{R} \setminus \{0\}} \frac{|f(x)|}{|x|} = \overline{f}, \qquad \sup_{x \in \mathbb{R} \setminus \{0\}} \frac{|g(x)|}{|x|} = \overline{g}.$$

Under conditions (2.3), (2.4), and the local Lipschitz continuity of f, g, (2.2) has a unique solution on any compact interval $[0, T]$ in the space of Itô processes (which is consequently continuous). In addition, the solution satisfies

$$\mathbb{E}\left[\max_{0\leq T}|X(t)|^2\right] < \infty.$$

These results are proven in Berger and Mizel [5]. To emphasise dependence on the initial condition, we denote the value of the solution of (2.2) at time $t \geq 0$ with initial condition ξ by $X(t;\xi)$. Moreover, by dint of (2.3), $X(t;0) = 0$ for all $t \geq 0$, a.s. This is called the zero solution of (2.2). It is the almost sure asymptotic stability of this solution which is the main topic of this paper. We say that all solutions are almost surely exponentially stable, or that the zero solution of (2.2) is globally almost surely exponentially stable, if there exists a positive nonrandom constant δ_0 such that for all ξ we have

$$\limsup_{t\to\infty}\frac{1}{t}\log|X(t,\xi)| \leq -\delta_0, \quad \text{a.s.} \tag{2.5}$$

In order to compare the stability of the solution of (2.2) with that of the deterministic analogue of (2.2)

$$x'(t) = -af(x(t)) + \int_0^t \lambda e^{-\mu(t-s)}g(X(s))\,ds, \quad t \geq 0 \tag{2.6}$$

(viz., (2.2) with $\sigma = 0$), we will wish to consider the linearisation of (2.6) at $x = 0$. We therefore assume that f and g are continuously differentiable in an open interval around 0, and without loss, we set

$$f'(0) = 1, \quad g'(0) = 1. \tag{2.7}$$

Note that by writing

$$y(t) = \int_0^t e^{-\mu(t-s)}g(x(s))\,ds,$$

we can re-express (2.6) as a first-order system of differential equations: if $z(t) = (x(t), y(t))$, then $z'(t) = F(z(t))$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F(x,y) = (-af(x) + \lambda y, g(x) - \mu y).$$

By considering the linearisation of this planar system at $x = 0$, we can use the Hartman-Grobman theorem to conclude that there exists η, $\delta > 0$ such that for all $|x(0)| < \delta$ implies

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| = -\delta, \tag{2.8}$$

whenever

$$a > \frac{\lambda}{\mu}, \quad a + \mu > 0. \tag{2.9}$$

Moreover, the zero solution of (2.6) is unstable if the sense of either (or both) of the inequalities in (2.9) is reversed.

In this paper, we wish to obtain conditions on σ, and the parameters a, μ, λ such that the solution of (2.2) is globally a.s. exponentially asymptotically stable (viz., that (2.5) above is satisfied for all initial conditions ξ and some $\delta_0 > 0$). In particular, we would like to establish (2.5) when (2.9) is false, establishing the asymptotic stability of (2.2) when the zero solution of (2.6) is unstable.

For much of the paper, we impose the following additional conditions on f:

$$xf(x) > 0, \quad \text{for all } x \neq 0, \tag{2.10}$$

and

$$0 < \underline{f} = \inf_{x\in\mathbb{R}\setminus\{0\}}\frac{|f(x)|}{|x|} \leq \sup_{x\in\mathbb{R}\setminus\{0\}}\frac{|f(x)|}{|x|} = \overline{f} < \infty. \tag{2.11}$$

Under (2.10), (2.11), more precise results on stabilisation are possible. We note that the linear case, in which $f(x) = x$ satisfies these conditions, and therefore use (2.10), (2.11) to obtain sharp results in this case. Moreover, once the analysis has been conducted under these conditions, it is easy to see how to proceed when they do not hold.

In studying the problem (2.1) where $|k(t)| \le \lambda e^{-\mu t}$ for some $\lambda, \mu > 0$, we will invoke the above hypotheses on f and g. We defer any further specific comments relating to this problem until we study it in Section 6, save to say that we try, in as far as is possible, to implement the program outlined above for (2.2) for the more general equation (2.1) also.

3. Exponential Asymptotic Stability

In this section, we establish a sufficient algebraic condition under which solutions of (2.2) satisfy (2.5). The analysis of this algebraic condition is the subject of Section 4.

Introduce the functions

$$\tilde{f}(x) = \begin{cases} f(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases} \tag{3.1}$$

and

$$\tilde{g}(x) = \begin{cases} g(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases} \tag{3.2}$$

Then by (2.3), (2.7), $\tilde{f}$, $\tilde{g}$ are continuous, and, by virtue of (2.4), bounded on $\mathbb{R}$, with $\sup_{y\in\mathbb{R}} |\tilde{g}(y)| = \overline{g}$. Next, introduce the process

$$Y(t) = \int_0^t e^{-\mu(t-s)} g(X(s))\, ds, \tag{3.3}$$

and define the matrices

$$A(y) = \begin{pmatrix} -a\tilde{f}(y) & \lambda \\ \tilde{g}(y) & -\mu \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

Then, setting

$$Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \tag{3.5}$$

we see that (3.1), (3.2), (3.3), (3.4) imply

$$dZ(t) = A(X(t))Z(t)\, dt + \Sigma Z(t)\, dB(t), \tag{3.6}$$

with $Z(0) = (\xi, 0)$. Next, suppose that P is an invertible matrix in $M_{2,2}(\mathbb{R})$, and define, for $z \in \mathbb{R}^2$

$$\tilde{A}(z) = PA(\langle P^{-1}z, \mathbf{e}_1\rangle)P^{-1}, \tilde{\Sigma} = P\Sigma P^{-1}, \tag{3.7}$$

and $\tilde{Z}(t) = PZ(t)$ so that $\tilde{Z}$ obeys the stochastic differential equation

$$d\tilde{Z}(t) = \tilde{A}(\tilde{Z}(t))\tilde{Z}(t)\, dt + \tilde{\Sigma}\tilde{Z}(t)\, dB(t), \tag{3.8}$$

and $\tilde{Z}(0) = PZ(0)$ is non-zero provided $\xi \neq 0$. Noticing from (3.4), (3.7) that $\tilde{A} \in C(\mathbb{R}^2; M_{2,2}(\mathbb{R}))$ is a bounded function, we can use Proposition 2.1 in Mao [8] to establish for $\xi \neq 0$ that

$$\tilde{Z}(t) \neq 0 \quad \text{for all } t \ge 0, \text{ a.s.} \tag{3.9}$$

Next, the boundedness of the entries of $\tilde{A}$ ensures the existence of a finite δ_0 such that

$$-\delta_0 = \sup_{y\in\mathbb{R}^2} \max_{\|x\|=1} \langle x, \tilde{A}(y)x\rangle + \frac{1}{2}\langle \tilde{\Sigma}x, \tilde{\Sigma}x\rangle - \langle x, \tilde{\Sigma}x\rangle^2. \tag{3.10}$$

Let us temporarily assume that it has been shown that

$$\delta_0 > 0. \tag{3.11}$$

Thus, for every $\delta_1 \in (0, \delta_0)$, there exists $p \in (0,1)$ satisfying

$$p \sup_{\|x\|=1} \langle \tilde{\Sigma} x, x\rangle^2 \leq 2(\delta_0 - \delta_1). \tag{3.12}$$

Define

$$p^* = 2\delta_0 / \sup_{\|x\|=1} \langle \tilde{\Sigma} x, x\rangle^2, \tag{3.13}$$

so that $p < p^*$ satisfies (3.12) for some $\delta_1 = \delta_1(p) > 0$.

By (3.10), (3.12), it follows for all $y \in \mathbb{R}^2$, $\|x\| = 1$ that

$$\langle x, \tilde{A}(y)x\rangle + \frac{1}{2}\langle \tilde{\Sigma} x, \tilde{\Sigma} x\rangle + \frac{p-2}{2}\langle x, \tilde{\Sigma} x\rangle^2 \leq -\delta_1. \tag{3.14}$$

Now, by (3.8), Itô's rule furnishes us with the semimartingale decomposition of $\|\tilde{Z}\|^2 = \{\|\tilde{Z}(t)\|^2\,; \mathcal{F}_t; 0 \leq t < \infty\}$, and so, by (3.9), we may use Itô's rule again to obtain the semimartingale decomposition of $\|\tilde{Z}\|^p = (\|\tilde{Z}\|^2)^{\frac{p}{2}}$, which is

$$\begin{aligned} \|\tilde{Z}(t)\|^p &= \|\tilde{Z}(0)\|^p \\ &\quad + \int_0^t p\|\tilde{Z}(s)\|^p \left(\frac{\langle \tilde{Z}(s), \tilde{A}(\tilde{Z}(s))\tilde{Z}(s)\rangle}{\|\tilde{Z}(s)\|^2} + \frac{1}{2}\frac{\langle \tilde{\Sigma}\tilde{Z}(s), \tilde{\Sigma}\tilde{Z}(s)\rangle}{\|\tilde{Z}(s)\|^2} \right. \\ &\qquad \left. + \frac{p-2}{2}\left(\frac{\langle \tilde{Z}(s), \tilde{\Sigma}\tilde{Z}(s)\rangle}{\|\tilde{Z}(s)\|^2}\right)^2 \right) ds \\ &\qquad + \int_0^t p\|\tilde{Z}(s)\|^{p-2}\langle \tilde{Z}(s), \tilde{\Sigma}\tilde{Z}(s)\rangle\, dB(s). \end{aligned} \tag{3.15}$$

Therefore, if t, $t+h \geq 0$, (3.14) and (3.15) yield

$$\begin{aligned} \|\tilde{Z}(t+h)\|^p - \|\tilde{Z}(t)\|^p &\leq \int_t^{t+h} -\delta_1 p\|\tilde{Z}(s)\|^p\, ds \\ &\qquad + \int_t^{t+h} p\|\tilde{Z}(s)\|^{p-2}\langle \tilde{Z}(s), \tilde{\Sigma}\tilde{Z}(s)\rangle\, dB(s). \end{aligned} \tag{3.16}$$

Using the Cauchy-Schwarz inequality, and then Liapunov's inequality (as $p < 1$), we get

$$\mathbb{E}\left[\int_0^t \|\tilde{Z}(s)\|^{2p-4}\langle \tilde{\Sigma}\tilde{Z}(s), \tilde{Z}(s)\rangle^2\, ds\right] \leq \|\tilde{\Sigma}\|^2 \int_0^t \mathbb{E}[\|\tilde{Z}(s)\|^2]^{1/p}\, ds$$

which is finite for all finite $t \geq 0$, as $\mathbb{E}[\|\tilde{Z}(t)\|^2]$ is finite for all $0 \leq t < \infty$. Therefore

$$\mathbb{E}\left[\int_0^t p\|\tilde{Z}(s)\|^{p-2}\langle \tilde{Z}(s), \tilde{\Sigma}\tilde{Z}(s)\rangle\, dB(s)\right] = 0,$$

so defining $V(t) = \mathbb{E}[\|\tilde{Z}(t)\|^p]$ and taking expectations across (3.16) yields

$$V(t+h) - V(t) \leq -\delta_1 p \int_t^{t+h} V(s)\, ds. \tag{3.17}$$

Obviously, $V(t)$ is nonnegative and finite for $0 \leq t < \infty$. In Lemma 3.4, we show that $t \mapsto V(t)$ is continuous. Therefore, taking the limsup as $h \to 0$ in (3.17), we arrive at

$$D_+ V(t) \leq -\delta_1 p V(t), \quad t \geq 0,$$

where

$$D_+V(t) = \limsup_{h\downarrow 0} \frac{V(t+h)-V(t)}{h}.$$

Therefore we get

$$\mathbb{E}[\|\tilde{Z}(t)\|^p] \le \mathbb{E}[\|\tilde{Z}(0)\|^p]e^{-\delta_1 pt}. \tag{3.18}$$

Using (3.5), the invertibility of P, and the definition of $\tilde{Z}$ in terms of Z, there exists a P-dependent constant $K(P)$, such that

$$\mathbb{E}[|X(t)|^p] \le K(P)\mathbb{E}[|\xi|^p]e^{-\delta_1 pt}, \quad t \ge 0. \tag{3.19}$$

This gives us our first result.

Theorem 3.1. *Suppose that there exists an invertible matrix $P \in M_{2,2}(\mathbb{R})$ such that δ_0 defined in (3.10) is positive. If p^* is defined by (3.13), then for $p < p^* \wedge 1$, there exists $\delta_1 = \delta_1(p) > 0$ such that (3.19) holds.*

To establish a.s. global exponential asymptotic stability, observe that the stochastic differential equation (3.8) satisfies all the hypotheses of Theorem 4.2 in Mao [10], so using this result, we get

$$\limsup_{t\to\infty} \frac{1}{t}\log\|\tilde{Z}(t)\| \le \frac{-\delta_1 p}{p} = -\delta_1, \quad \text{a.s.}$$

Letting $\delta_1 \uparrow \delta_0$ through the rationals yields

$$\limsup_{t\to\infty} \frac{1}{t}\log\|\tilde{Z}(t)\| \le -\delta_0, \quad \text{a.s.}$$

and so, using $\tilde{Z}(t) = PZ(t)$ and (3.5) now gives

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t)| \le -\delta_0, \quad \text{a.s.}$$

We therefore have the following theorem.

Theorem 3.2. *Suppose that there exists an invertible matrix $P \in M_{2,2}(\mathbb{R})$ such that δ_0 defined by (3.10) is positive. Then the solution of (2.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t;\xi)| \le -\delta_0, \quad \textit{a.s.}$$

Hence, if δ_0 defined by (3.10) can be shown to be positive, Theorems 3.1 and 3.2 show that all solutions of (2.2) are p-th mean exponentially stable for some $p \in (0,1)$, and also are a.s. globally exponentially asymptotically stable.

Remark 3.3. We could have proved the almost sure exponential stability under the condition that δ_0 defined by (3.10) is positive more directly by obtaining a semi-martingale decomposition of $\log(X(t)^2 + Y(t)^2)$. This is essentially Mao's method of proving stabilisation of ordinary differential equations by noise in Theorem 3.1 of [9]. However, as we prefer to also establish the p-th mean exponential asymptotic stability for small p, we proceed as in Theorems 3.1, 3.2 above.

We now return to the deferred proof of Lemma 3.4.

Lemma 3.4. *If $\tilde{Z}$ satisfies (3.8), then $t \mapsto \mathbb{E}[\|\tilde{Z}(t)\|^p]$ is continuous for any $p \in (0,1)$.*

Proof. Since $\tilde{A}$ is a bounded function, Problem 5.3.15 in Karatzas and Shreve [7] implies that $\tilde{Z}$ satisfies

$$\mathbb{E}[\|\tilde{Z}(t)-\tilde{Z}(t_0)\|^p] \le C(T)\left(1+\mathbb{E}[\|\tilde{Z}(0)\|^2]\right)|t-t_0| \tag{3.20}$$

for all $0 \le t_0, t \le T$, and any $T > 0$. The constant $C(T)$ is positive and finite. Note that $p \in (0,1)$, so

$$\begin{aligned} &|\mathbb{E}\|Z(t)\|^p - \mathbb{E}\|Z(t_0)\|^p| \\ &= |\mathbb{E}[\|Z(t)\|^p - \|Z(t_0)\|^p]| \le \mathbb{E}\,|\|Z(t)\|^p - \|Z(t_0)\|^p| \\ &\le \mathbb{E}\,[|\|Z(t)\| - \|Z(t_0)\||^p] \le \mathbb{E}\left[|\|Z(t)\| - \|Z(t_0)\||^2\right]^{p/2}, \end{aligned}$$

where we use the inequality $|x^p - y^p| \le |x-y|^p$, $x, y \ge 0$ at the penultimate step, and Liapunov's inequality at the last step. Applying the inequality $|\|x\| - \|y\|| \le \|x-y\|$ for $x, y \in \mathbb{R}^2$, and then (3.20) therefore gives

$$\left|\mathbb{E}\|\tilde{Z}(t)\|^p - \mathbb{E}\|\tilde{Z}(t_0)\|^p\right| \le \left(C(T)(1 + \mathbb{E}[\|\tilde{Z}(0)\|^2])|t - t_0|\right)^{p/2},$$

from which continuity is immediate. □

4. Sufficient conditions for $\delta_0 > 0$

In this section, we obtain some explicit conditions in terms of the parameters in (2.2) for which δ_0 given by (3.10) can be shown to be positive. In most cases, we do not try to calculate δ_0 directly, but instead obtain bounds on the underlying quantity

$$F(x, y) = \langle x, \tilde{A}(y)x\rangle + \frac{1}{2}\langle\tilde{\Sigma}x, \tilde{\Sigma}x\rangle - \langle x, \tilde{\Sigma}x\rangle^2 \tag{4.1}$$

which is to be maximised. To do this, we consider a simple class of diagonal matrices P

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \tag{4.2}$$

for $\beta > 0$ in the construction of the matrices $\tilde{A}$, $\tilde{\Sigma}$ in (3.7). We will try to choose β optimally. Defining $x = (x_1, x_2)$, $w = \langle P^{-1}y, \mathbf{e}_1\rangle$, and using (3.4), (3.7), (4.2), we get

$$\begin{aligned} F(x, y) = -a\tilde{f}(w)x_1^2 + \left(\frac{\lambda}{\beta} + \beta\tilde{g}(w)\beta\right) x_1x_2 \\ - \mu x_2^2 - \sigma^2 x_1^4 + \frac{1}{2}\sigma^2 x_1^2 =: \tilde{F}(x, w). \end{aligned} \tag{4.3}$$

Next define

$$\Lambda(\beta) = \sup_{w\in\mathbb{R}} \left|\frac{\lambda}{\beta} + \beta\tilde{g}(w)\beta\right|. \tag{4.4}$$

Then there exists $\beta^* > 0$ such that

$$\Lambda^* := \Lambda(\beta^*) = \inf_{\beta>0} \Lambda(\beta). \tag{4.5}$$

We will give explicit formulae for such a Λ^* presently. Returning to (4.3) with $\beta = \beta^*$, and by using the inequality

$$x_1x_2 \le \frac{1}{2}(\alpha x_1^2 + \frac{1}{\alpha}x_2^2), \quad \alpha > 0,$$

we obtain

$$\tilde{F}(x, w) \le -a\tilde{f}(w)x_1^2 + \frac{1}{2}\Lambda^*\left(\alpha x_1^2 + \frac{1}{\alpha}x_2^2\right) - \mu x_2^2 - \sigma^2 x_1^4 + \frac{1}{2}\sigma^2 x_1^2. \tag{4.6}$$

We reimpose conditions (2.10), (2.11). By defining

$$\tilde{f} = \begin{cases} \inf_{w\in\mathbb{R}} \tilde{f}(w) = \underline{f}, & \text{for } a \ge 0 \\ \sup_{w\in\mathbb{R}} \tilde{f}(w) = \overline{f}, & \text{for } a < 0 \end{cases}, \tag{4.7}$$

we see from (3.10), (4.6), and (4.7) that

$$-\delta_0 \le \max_{x\in[0,1]} H(x) \tag{4.8}$$

where

$$H(x) = -\sigma^2 x^2 + x\left(-a\tilde{f} + \frac{1}{2}\Lambda^*(\alpha - \alpha^{-1}) + \mu + \frac{1}{2}\sigma^2\right) + \frac{\Lambda^*}{2\alpha} - \mu. \tag{4.9}$$

Note that σ enters (4.9) only through terms involving σ^2, so that without loss, we consider $\sigma > 0$ hereinafter.

The effect of the above argument has been to bound the complicated nonlinear function F with a one-parameter family of quadratic functions. Therefore, in order to show that $\delta_0 > 0$ is possible for some interval of values of σ for a particular triple of parameters (μ, a, Λ^*) (determined by the deterministic problem (2.6)), it is sufficient to prove that there is a value of $\alpha > 0$ — which can depend on (μ, a, Λ^*) — such that for that choice of α, there is an interval of values of $|\sigma|$ such that

$$\max_{x\in[0,1]} H(x) < 0. \tag{4.10}$$

Remark 4.1. We can separate out the dependence of Λ^* on λ by observing (for $\lambda > 0$) that

$$\Lambda^* = \inf_{\beta>0} \Lambda(\sqrt{\lambda}\beta) = \sqrt{\lambda} \inf_{\beta>0} \sup_{w\in\mathbb{R}} |\beta^{-1} + \beta\tilde{g}(w)|,$$

and similarly for $\lambda < 0$ that

$$\Lambda^* = \inf_{\beta>0} \Lambda(\sqrt{-\lambda}\beta) = \sqrt{-\lambda} \inf_{\beta>0} \sup_{w\in\mathbb{R}} |-\beta^{-1} + \beta\tilde{g}(w)|,$$

so $\Lambda^*(\lambda) = \sqrt{|\lambda|}G^*(\lambda)$, where $G^*(\lambda) = \inf_{\beta>0}\sup_{w\in\mathbb{R}} |\operatorname{sgn}(\lambda)\beta^{-1} + \beta\tilde{g}(w)|$. In fact, it is possible to explicitly compute $\Lambda^*(\lambda)$, given $g_0 = \inf_{w\in\mathbb{R}} \tilde{g}(w)$ and $g^0 = \sup_{w\in\mathbb{R}} \tilde{g}(w)$. For $\lambda > 0$ we have

$$\Lambda^*(\lambda) = \begin{cases} 2\sqrt{g^0}\sqrt{\lambda} & \text{if } g_0 > 0 \text{ or } 0 < -g_0/3 < g^0 \\ \sqrt{\lambda}\frac{1}{\sqrt{2}}\frac{-g_0+g^0}{\sqrt{-g_0-g^0}} & \text{if } -g_0/3 > g^0, \end{cases}$$

while for $\lambda < 0$, Λ^* is given by

$$\Lambda^*(\lambda) = \begin{cases} \sqrt{|\lambda|}\frac{1}{\sqrt{2}}\frac{g^0-g_0}{\sqrt{g^0+g_0}} & \text{if } g_0 > 0 \text{ or } -g_0 > g^0/3 > 0 \\ 2\sqrt{|\lambda|}\sqrt{-g_0} & \text{if } -g_0 > g^0/3. \end{cases}$$

By a careful choice of α, the following can be proven:

Proposition 4.2. *Suppose (2.10), (2.11) are true, and define*

$$\nu_\lambda(\mu) = \begin{cases} \frac{\Lambda^*(\lambda)^2}{4f}\frac{1}{\mu} & \text{if } 0 < \mu \le \frac{\Lambda^*(\lambda)}{2}, \\ \frac{\Lambda^*(\lambda)-\mu}{f} & \text{if } \frac{\Lambda^*(\lambda)}{2} \le \mu \le \Lambda^*(\lambda), \\ \frac{\Lambda^*(\lambda)-\mu}{\overline{f}} & \text{if } \mu > \Lambda^*(\lambda), \end{cases} \tag{4.11}$$

where Λ^* is given by (4.5). If

$$a > \nu_\lambda(\mu), \tag{4.12}$$

there exists a non-empty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\lambda}$, there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda) > 0$ so that

$$\delta_0^* < \delta_0, \tag{4.13}$$

where δ_0 is given by (3.10).

Proof. In the proof, we suppress the dependence of Λ^* on λ. Define

$$\mu_1 = \frac{\Lambda^*}{2\alpha}, \quad \mu_2 = \frac{1}{2}\Lambda^*(\alpha + \alpha^{-1}), \quad \mu_3 = \Lambda^*\left(\alpha + \frac{1}{2\alpha}\right) \tag{4.14}$$

and the functions

$$f_1(\mu) = \frac{\mu + \frac{1}{2}\Lambda^*(\alpha - \alpha^{-1})}{\underline{f}}, \quad \mu \geq \mu_1, \tag{4.15}$$

$$f_2(\mu) = \begin{cases} \frac{\Lambda^*(2\alpha+\alpha^{-1})-2\mu}{4\underline{f}}, & \mu_1 \leq \mu \leq \mu_2 \\ \frac{\Lambda^*(2\alpha+\alpha^{-1})-2\mu}{4\overline{f}}, & \mu_2 \leq \mu \end{cases} \tag{4.16}$$

$$f_3(\mu) = \begin{cases} \frac{\frac{1}{2}\Lambda^*(\alpha+\alpha^{-1})-\mu}{4\underline{f}}, & \mu_2 \leq \mu \leq \mu_3 \\ \frac{\frac{1}{2}\Lambda^*(\alpha+\alpha^{-1})-\mu}{4\overline{f}}, & \mu_3 \leq \mu \end{cases} \tag{4.17}$$

Note that

$$f_1(\mu_1) = f_2(\mu_1) = f_3(\mu_1) = \frac{\alpha\Lambda^*}{2\underline{f}}. \tag{4.18}$$

There are three ways in which (4.10) can be satisfied for H defined in (4.9)

(i) $H(0) < 0$, $H'(0) < 0$ (so $\max_{x\in[0,1]} H(x) = H(0) < 0$);
(ii) $H(1) < 0$, $H'(1) > 0$ (so $\max_{x\in[0,1]} H(x) = H(1) < 0$);
(iii) There exists $x^* \in [0,1]$ with $H'(x^*) = 0$, $H(x^*) < 0$ (so $\max_{x\in[0,1]} H(x) = H(x^*) < 0$).

Consider (for fixed α, λ) the subdivision of the (μ, a)-plane

(1) $a > f_1(\mu)$, $\mu > \mu_1$,
(1') $a = f_1(\mu)$, $\mu > \mu_1$,
(2) $a < f_1(\mu)$, $a > f_2(\mu)$, $\mu > \mu_1$,
(3) $a \leq f_2(\mu)$, $a > f_3(\mu)$, $\mu > \mu_1$.

It obtains after some manipulation that (1) implies (i), (2) implies (ii), (3) implies (iii) and (1') implies (iii) (with $x^* = 1/4$) for appropriate ranges of σ. For future reference, we note that the ranges of σ are given by

$$\frac{1}{2}\sigma^2 < \overline{f}(a - f_1(\mu)) \quad \text{for (1)}, \tag{4.19}$$

$$\sigma^2 < 16(\mu - \mu_1) \quad \text{for (1')}, \tag{4.20}$$

$$2(-a\tilde{f} + \frac{\Lambda^*\alpha}{2}) < \sigma^2 < \frac{2}{3}(-a\tilde{f} + \mu + \frac{\Lambda^*}{2}(\alpha - \alpha^{-1}) \quad \text{for (2)}, \tag{4.21}$$

and

$$\sigma_- < |\sigma| < \sigma_+, \quad \text{for (3)} \tag{4.22}$$

where

$$\sigma_\pm = 2\sqrt{\mu - \mu_1} \pm \sqrt{2\tilde{f}(a - f_3(\mu))}. \tag{4.23}$$

Fix $\alpha > 0$, $\Lambda^* > 0$ (by fixing λ). Then with

$$S_\alpha = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R} \,:\, \mu > \frac{\Lambda^*}{2\alpha},\, a > f_3(\mu)\}, \tag{4.24}$$

the following is true:

if $(\mu, a) \in S_\alpha$ for some $\alpha > 0$, there exists a non-empty interval $I_{a,\mu,\lambda,\alpha} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\lambda,\alpha}$ we have $H(x) < 0$ for all $x \in [0,1]$. (The intervals $I_{a,\mu,\lambda,\alpha}$ in cases (1), (1'), (2), (3) are determined by (4.19), (4.20), (4.21), (4.22), respectively.)

Suppose $\nu_\lambda(\mu)$ is as defined in (4.11), and (4.12) holds. Then for $\mu \geq \Lambda^*$, $(\mu, a) \in S_1$, and for $\frac{\Lambda^*}{2} \leq \mu < \Lambda^*$, $(\mu, a) \in S_1$. For $0 < \mu < \frac{\Lambda^*}{2}$, $a > \nu_\lambda(\mu)$, we get $a > 0$, so there exists $\alpha > 0$ such that

$$\frac{\Lambda^*}{2\mu} < \alpha < \frac{2\underline{f}a}{\Lambda^*},$$

which gives $a > \alpha\Lambda^*/2\underline{f}$, $\mu > \Lambda^*/2\alpha$, so by (4.18), (4.24), we have $(\mu, a) \in S_\alpha$. Fixing the α-dependence in terms of a, μ, λ as above yields the appropriate intervals $I_{a,\mu,\lambda} := I_{a,\mu,\lambda,\alpha(a,\mu,\lambda)}$, where $-\delta_0^* = \max_{x\in[0,1]} H(x)$ satisfies (4.13), by (4.8). □

Remark 4.3. Notice that $\Lambda^* = 0$ if and only if $g(x) \equiv x$, and $\lambda < 0$, in which case, by defining

$$\nu_\lambda(\mu) = -\frac{\mu}{\underline{f}}, \quad \mu > 0,$$

the result of Proposition 4.2 is true where $a > \nu_\lambda(\mu)$. Note that the formula for ν_λ coincides with that in (4.11) when $\Lambda^*(\lambda) = 0$.

5. Exponential Kernel

This section contains the first set of principal results of this paper. The first subsection states the main results flowing from the analysis in the previous two sections, the second considers the stabilisation of the nonlinear equation (2.6), while the third subsection deals with stabilisation and non-destabilising results relating to the linear version of (2.6).

5.1. **Theorems.** Combining the results of Theorems 3.1, and 3.2 , and of Proposition 4.2, we immediately obtain the first main result of the paper.

Theorem 5.1. *Suppose that f, g are locally Lipschitz continuous functions such that (2.3), (2.4), (2.7), (2.10), (2.11) hold. Let ν_λ be defined by (4.11) and $a > \nu_\lambda(\mu)$. Then there exists a non-empty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$ there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda) > 0$ so that all solutions of (2.2) satisfy*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t;\xi)| \leq -\delta_0^*, \quad a.s. \tag{5.1}$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there is $\delta_p > 0$ such that

$$\mathbb{E}[|X(t;\xi)|^p] \leq K\mathbb{E}[|\xi|^p|]e^{-\delta_p t}, \quad t \geq 0, \tag{5.2}$$

where K is a positive constant independent of p and ξ.

The analysis of Proposition 4.2 enables a stabilisation result to be proven when conditions (2.10), (2.11) on f are dropped. In that case, we still have (4.8), but with

$$H(x) = |a|\overline{f}x + \frac{1}{2}\Lambda^*(\alpha x + \alpha^{-1}(1-x)) - \mu(1-x) - \sigma^2 x^2 + \frac{1}{2}\sigma^2 x,$$

as $-a\tilde{f}(w) \leq |a|\overline{f}$, by (2.4), (3.1). Hence, by replacing $-a$ by $|a|$ and $\tilde{f}$ by $\overline{f}$ in Proposition 4.2, we have the following result.

Theorem 5.2. *Suppose that f, g are locally Lipschitz continuous functions such that (2.3), (2.4), (2.7) hold. Let Λ^* be defined by (4.5), and suppose*

$$|a| < \frac{\mu - \Lambda^*(\lambda)}{\overline{f}}, \quad \mu > \Lambda^*(\lambda).$$

Then there exists a non-empty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\lambda}$ there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda) > 0$ so that all solutions of (2.2 satisfy

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t;\xi)| \leq -\delta_0^*, \quad a.s.$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there is $\delta_p > 0$ such that

$$\mathbb{E}[|X(t;\xi)|^p] \leq K\mathbb{E}[|\xi|^p|]e^{-\delta_p t}, \quad t \geq 0$$

where K is a positive constant independent of p and ξ.

Remark 5.3. It does not appear that conditions given in Theorem 3.1 in Mao [9] can be applied directly to the example given here, owing to the structure of the matrix Σ in (3.4). Moreover, it does not seem that apply a co-ordinate transform will rectify this.

Remark 5.4. It appears that it is possible to exploit the method of proof of Theorem 4 in [3] to obtain results on the stabilisation of (2.6) by noise. However, the estimates obtained on the region in (μ, a, λ) parameter space in which we can state with certainty that all solutions of (2.2) tend to zero are less sharp than those obtained in Theorem 5.1, 5.2, under the hypotheses imposed in those theorems.

Remark 5.5. Under the hypotheses (2.10), (2.11) with $a > 0$, it can be shown that

$$a\underline{f} > \frac{\lambda\overline{g}}{\mu} \tag{5.3}$$

implies that all solutions of (2.2) satisfy (5.1) for some $\delta_0 > 0$, for *all* values of $|\sigma| > 0$. The proof of this result in the linear case is established in [2]; the adaptations required in the nonlinear case are straightforward, and not recorded here.

5.2. **Stabilisation of the nonlinear equation (2.6).** We consider the deterministic nonlinear equation (2.6) and the corresponding family of related Itô-Volterra equations (2.2), under the hypotheses (2.10), (2.11). We deal with the cases $\lambda > 0$, $\lambda < 0$ separately.

For $\lambda > 0$, the zero solution of (2.6) is locally exponentially asymptotically stable if $(\mu, a, \lambda) \in S_D$, where

$$S_D = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R}^+ \ : \ a > \frac{\lambda}{\mu}\}$$

For (2.2), the zero solution is almost surely exponentially asymptotically stable if $(\mu, a, \lambda) \in S_S$

$$S_S = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R}^+ \ : \ a > \nu_\lambda(\mu)\}$$

if $|\sigma| \in I_{a,\mu,\lambda}$. By defining $\mu^*(\lambda) \geq \Lambda(\lambda)/2$ as the solution of $\lambda/\mu^*(\lambda) = \nu_\lambda(\mu^*(\lambda))$, we see that for all $\mu > \mu^*(\lambda)$ that for all

$$\frac{\lambda}{\mu} < a < \nu_\lambda(\mu) \tag{5.4}$$

there exists an interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$ the deterministic problem (2.6) with parameters (μ, a, λ) is unstable, while problem (2.2) with parameters $(\mu, a, \lambda, \sigma)$ is globally asymptotically stable, a.s.

For $\lambda < 0$, using the same notation as above, we have

$$S_D = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R}^+ \ : \ a > \frac{\lambda}{\mu},\ a + \mu > 0\},$$
$$S_S = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R}^+ \ : \ a > \nu_\lambda(\mu)\}.$$

Hence, by defining $\mu^*(\lambda) \geq \Lambda(\lambda)$ as above, with $\mu > \mu^*(\lambda)$, and a obeying (5.4), the deterministic problem is unstable, while the stochastic problem can be stabilised for the appropriate choice of σ, as above.

Relaxing the hypotheses (2.10), (2.11), we see that the unstable deterministic problem with parameters (μ, a, λ) can be stabilised for any $|\sigma| \in I_{a,\mu,\lambda}$, which is a non-empty interval in $\mathbb{R}^+$ provided

$$a < \frac{\lambda}{\mu}, \quad |a| < \frac{\mu - \Lambda^*(\lambda)}{\overline{f}}, \quad \mu > \Lambda^*(\mu).$$

5.3. **Stabilisation of the linear equation.** For a given value of λ, we explicitly recognise the λ-dependence of S_D and S_S by writing $S_D = S_D(\lambda)$, $S_S = S_S(\lambda)$. Then for fixed λ it is the case that $\overline{S_S(\lambda)} \cap S_D(\lambda)$ and $S_S(\lambda) \cap \overline{S_D(\lambda)}$ are, in general, non-empty. This does not mean, however, that noise can also have a destabilising effect, as $(\mu, a, \lambda) \notin S_S$ does not imply that there is destabilisation. For the linear version of (2.2), (2.6) however (where $f(x) = g(x) = x$), we show in the this section that $\overline{S_S(\lambda)} \cap S_D(\lambda)$ is empty for all $\lambda \neq 0$, so that noise can always be added to a stable deterministic problem in such a way that the stochastic problem remains stable, while $S_S(\lambda) \cap \overline{S_D(\lambda)}$ is nonempty for all $\lambda \neq 0$, so that there exist unstable deterministic problems which can be stabilisied. It is therefore justifiable to talk about the stabilising effect of noise for linear problems.

To make the discussion concrete, the deterministic linear problem is

$$x'(t) = -ax(t) + \int_0^t \lambda e^{-\mu(t-s)} x(s)\, ds \tag{5.5}$$

with the corresponding family of Itô-Volterra equations

$$dX(t) = \left(-aX(t) + \int_0^t \lambda e^{-\mu(t-s)} X(s)\, ds\right) dt + \sigma X(t)\, dB(t). \tag{5.6}$$

Consider $\lambda > 0$, and note that we have

$$\nu_\lambda(\mu) = \begin{cases} \lambda/\mu & \text{if } 0 < \mu \leq \sqrt{\lambda} \\ 2\sqrt{\lambda} - \mu & \text{if } \sqrt{\lambda} < \mu \end{cases}$$

Therefore $S_D(\lambda) \subset S_S(\lambda)$ for all $\lambda > 0$, as claimed. For $\lambda < 0$, we get $\nu_\lambda(\mu) = -\mu$ for all $\mu > 0$, and again $S_D(\lambda) \subset S_S(\lambda)$ for all $\lambda < 0$.

The method outlined in this paper for obtaining upper bounds on the a.s. decay rate of Itô-Volterra equations of the form (2.2) is sharp for the linear problem (5.6) when $\lambda > 0$ (and $\sigma = 0$) in the following sense: the region in (μ, a, λ)-parameter space for which convergence occurs is correctly identified ($a > \lambda/\mu$), and the exponential decay rate of solutions identified as a solution of $\gamma^2 + (a+\mu)\gamma + (a\mu - \lambda) = 0$.

To see this, note for $\sigma = 0$, $\lambda > 0$ that $H(x) = x(-a + \sqrt{\lambda}(\alpha - \alpha^{-1}) + \mu) + (-\mu + \sqrt{\lambda}\alpha^{-1})$. Then $H(0)$, $H(1) < 0$ if and only if we can choose $\alpha \in (\sqrt{\lambda}/\mu, a/\sqrt{\lambda})$, which requires $a > \lambda/\mu$. Define the bound on the decay rate by $\gamma = \sup_{x \in [0,1]} H(x) = H(0) \vee H(1)$. The best estimate can be obtained by choosing α such that $H(0) = H(1)$, in which case we get the required $\gamma^2 + (a+\mu)\gamma + (a\mu - \lambda) = 0$.

6. General Exponential Kernel

We now study the general scalar Itô-Volterra equation (2.1), relaxing the hypothesis that the kernel k is exponential, instead assuming that

$$|k(t)| \leq \lambda e^{-\mu t}, \quad t \geq 0. \tag{6.1}$$

for some $\lambda, \mu > 0$. In this case, the solution of the Itô-Volterra equation cannot be written in terms of a stochastic differential equation. However, the condition (6.1) enables us to be able to obtain an upper bound on solutions of (2.1) in terms of a linear problem of the form (5.6), from which we can obtain the required bound on the negative Liapunov exponent.

Lemma 6.1. *Suppose that f, g satisfy (2.3), (2.4), (2.7), (2.10), (2.11), k satisfies (6.1), and X is the solution of (2.1). Let $(Y(t))_{t\geq 0}$ be the process given by $Y(0) = |X(0)|$ and*

$$dY(t) = \left(-a\tilde{f}Y(t) + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}Y(s)\,ds\right)dt + \sigma Y(t)\,dB(t), \tag{6.2}$$

where $\tilde{f}$ is given by (4.7) and $\overline{g}$ by (2.4). Then $|X(t)| \leq Y(t)$ for all $t \geq 0$, a.s.

Proof. Define $(\phi(t))_{t\geq 0}$ by $\phi(0) = 1$ and $d\phi(t) = \sigma\phi(t)\,dB(t)$, and let $X_1(t) = \phi(t)^{-1}X(t)$. Using integration by parts, we find that

$$X_1(t) = X(0) + \int_0^t \phi(s)^{-1}R(s)\,ds,$$

where $R(t) = -af(\phi(t)X_1(t)) + \int_0^t k(t-s)g(\phi(s)X_1(s))\,ds$. Since R and ϕ have continuous paths, we see that X_1 is C^1, and

$$X_1'(t) = -af(\phi(t)X_1(t))\phi(t)^{-1} + \int_0^t k(t-s)g(\phi(s)X_1(s))\phi(t)^{-1}\,ds. \tag{6.3}$$

We now analyse this smooth system (6.3) on a pathwise basis, so we assume $\omega \in \Omega$ is fixed. Considering the case $a > 0$, and invoking (2.10), (2.11), we have

$$D_-|X_1(t)| \leq -a\underline{f}|X_1(t)| + \left|\int_0^t k(t-s)g(\phi(s)X_1(s))\phi(t)^{-1}\,ds\right|,$$

so with $a > 0$, and using (2.4), (4.7), (6.1) gives

$$D_-|X_1(t)| \leq -a\tilde{f}|X_1(t)| + \int_0^t \lambda e^{-\mu(t-s)}\overline{g}|X_1(s)|\phi(s)\phi(t)^{-1}\,ds. \tag{6.4}$$

The same analysis for $a \leq 0$ also gives (6.4), recalling that $\tilde{f}$ is defined by (4.7). Introduce the process $Y_1(t) = \phi(t)^{-1}Y(t)$, so $Y_1(0) = |X(0)|$, and (6.2) gives

$$Y_1'(t) = -a\tilde{f}Y(t) + \int_0^t \lambda e^{-\mu(t-s)}\overline{g}Y_1(s)\phi(s)\phi(t)^{-1}\,ds. \tag{6.5}$$

Applying the deterministic comparison principle on a pathwise basis now gives $|X_1(t)| \leq Y_1(t)$ for all $t \geq 0$, and almost all paths $\omega \in \Omega$. The result follows by the definition of X_1, Y_1. $\square$

Using Theorem 5.1, we can determine when the top Liapunov exponent of Y defined by (6.2) is negative, and hence obtain a stabilisation result for (2.1).

Theorem 6.2. *Let X be the solution of (2.1). Suppose that f, g satisfy (2.3), (2.4), (2.7), (2.10), (2.11), and k satisfies (6.1) for some λ, $\mu > 0$. Define*

$$\nu_\lambda(\mu) = \begin{cases} \frac{\lambda\overline{g}}{\mu\underline{f}} & \text{if } 0 < \mu \leq \sqrt{\lambda\overline{g}} \\ \frac{2\sqrt{\lambda\overline{g}}-\mu}{\underline{f}} & \text{if } \sqrt{\lambda\overline{g}} \leq \mu \leq 2\sqrt{\lambda\overline{g}} \\ \frac{2\sqrt{\lambda\overline{g}}-\mu}{\overline{f}} & \text{if } \mu > 2\sqrt{\lambda\overline{g}} \end{cases}$$

If $a > \nu_\lambda(\mu)$, there exists a non-empty interval $I_{a,\mu,\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda}$ there exists $\delta_0^* = \delta_0^*(a,\mu,\lambda,|\sigma|) > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t;\xi)| \leq -\delta_0^*, \quad a.s.$$

Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there is $\delta_p > 0$ such that

$$\mathbb{E}[|X(t;\xi)|^p] \leq K\mathbb{E}[|\xi|^p]]e^{-\delta_p t}, \quad t \geq 0$$

where K is a positive constant independent of p and ξ.

Removing the hypotheses (2.10), (2.11), we obtain a similar result to Lemma 6.1 using a nearly identical argument.

Lemma 6.3. *Suppose that f, g satisfy (2.3), (2.4), (2.7), k satisfies (6.1), and X is the solution of (2.1). Let $(Y(t))_{t\geq 0}$ be the process given by $Y(0)=|X(0)|$ and*

$$dY(t)=\left(|a|\overline{f}Y(t)+\int_0^t \lambda\overline{g}e^{-\mu(t-s)}Y(s)\,ds\right)dt+\sigma Y(t)\,dB(t), \tag{6.6}$$

where $\overline{f},\overline{g}$ are given by (2.4). Then $|X(t)|\leq Y(t)$ for all $t\geq 0$, a.s.

(6.6) is in the form of (5.6) with $-|a|\overline{f}<0$ in the role of a and $\lambda\overline{g}$ in the role of λ, so the stabilisation region in (μ,a)-parameter space for fixed λ is given by

$$-|a|\overline{f}>2\sqrt{\lambda\overline{g}}-\mu,\quad \mu>2\sqrt{\lambda\overline{g}}.$$

Therefore, we have

Theorem 6.4. *Let X be the solution of (2.1). Suppose that f, g satisfy (2.3), (2.4), (2.7), and k satisfies (6.1) for some λ, $\mu>0$. If*

$$|a|<\frac{2\sqrt{\lambda\overline{g}}-\mu}{\overline{f}},\quad \mu>2\sqrt{\lambda\overline{g}},$$

there exists a non-empty interval $I_{a,\mu,\lambda}\subset\mathbb{R}^+$ such that for $|\sigma|\in I_{a,\mu,\lambda}$ there exists $\delta_0^*=\delta_0^*(a,\mu,\lambda,|\sigma|)>0$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\log|X(t;\xi)|\leq-\delta_0^*,\quad a.s.$$

Moreover, there exists $p^*>0$ such that for all $p<p^*$, there is $\delta_p>0$ such that

$$\mathbb{E}[|X(t;\xi)|^p]\leq K\mathbb{E}[|\xi|^p]]e^{-\delta_p t},\quad t\geq 0$$

where K is a positive constant independent of p and ξ.

7. Examples

To consider the stabilisation of solutions of the deterministic equation

$$x'(t)=-af(x(t))+\int_0^t k(t-s)g(x(s))\,ds, \tag{7.1}$$

(where k satisfies (6.1)), we observe by (2.6) that the zero solution of (7.1) is unstable if

$$a<\int_0^\infty k(s)\,ds, \tag{7.2}$$

as this guarantees that the linearisation of (7.1) is unstable. As an example, suppose that $k(t)>0$ for all $t\geq 0$, so that for $a<0$ (7.1) is unstable. If, in addition $a>(2\sqrt{\lambda\overline{g}}-\mu)/\mu$ and $\mu>2\sqrt{\lambda\overline{g}}$, there exists an interval in $\mathbb{R}^+$ such that for all $|\sigma|\in I_{a,\mu,\lambda}$, (2.1) is a.s. globally exponentially asymptotically stable, so stabilisation is again possible.

We now give an example under which Theorem 6.2 applies

Example 7.1.

Consider the deterministic problem

$$x'(t)=-ax(t)(2-\cos(x(t)))+\int_0^t\frac{1}{1+(t-s)^2}e^{-\mu(t-s)}\left(x(s)+\frac{x(s)^3}{1+x(s)^2}\right)ds \tag{7.3}$$

with the corresponding family of stochastic Volterra equations

$$(7.4)\quad dX(t) = \Bigg(-aX(t)(2-\cos(X(t))) + \int_0^t \frac{1}{1+(t-s)^2} e^{-\mu(t-s)} \left(X(s) + \frac{X(s)^3}{1+X(s)^2} \right) ds \Bigg) dt + \sigma X(t)\, dB(t).$$

Therefore, we identify $f(x) = x(2-\cos x)$, $g(x) = x + x^3/(1+x^2)$, and note that f, g satisfy (2.3), (2.4), (2.7), (2.10), (2.11), and are both locally Lipschitz. We have $\overline{g} = 2$ and

$$\tilde{f} = \begin{cases} 1 & \text{if } a > 0, \\ 3 & \text{if } a \le 0. \end{cases}$$

Note that $k(t) = (1+(t^2+1)^{-1})e^{-\mu t}$ satisfies the condition (6.1) with $\lambda = 2$. Since

$$\int_0^\infty k(s)\, ds > \frac{1}{\mu},$$

we see that (7.4) is unstable for $a < 1/\mu$. Define

$$\nu(\mu) = \begin{cases} \frac{4}{\mu} & \text{if } 0 < \mu \le 2, \\ 4-\mu & \text{if } 2 < \mu \le 4, \\ \frac{1}{3}(4-\mu) & \text{if } \mu > 4. \end{cases}$$

Hence for $\mu > 1+\sqrt{3}$, $\nu(\mu) < a < 1/\mu$, Theorem 6.2 shows that all solutions of (7.4) tend to zero exponentially fast, almost surely, provided that $|\sigma|$ is contained in an open interval contained in $\mathbb{R}^+$ whose endpoints depend on a, μ. At the same time, the zero solution of the deterministic problem (7.3) is unstable.

We finally show how the estimates on the ranges of admissible σ for which (2.2) is stable in terms of the free parameter α can be established, using a specific example to highlight the construction.

Example 7.2.

Consider the nonlinear scalar deterministic Volterra equation

$$(7.5)\qquad x'(t) = x(t) - \frac{1}{2}\frac{x(t)^3}{1+x(t)^2} + \int_0^t 2e^{-10(t-s)}(x(s) + \sin x(s))\, ds,$$

and the corresponding family of Itô-Volterra equations

$$(7.6)\quad dX(t) = \left(X(t) - \frac{1}{2}\frac{X(t)^3}{1+X(t)^2} + \int_0^t 2e^{-10(t-s)}(X(s) + \sin X(s))\, ds \right) dt + \sigma X(t)\, dB(t).$$

Note that equation (7.6) is of the form (2.2) and satisfies the restrictions (2.3), (2.4), (2.7), (2.10), (2.11), where

$$f(x) = x - \frac{1}{2}\frac{x^3}{1+x^2}, \quad g(x) = \frac{1}{2}(x + \sin x),$$

and $\lambda = 4$, $a = -1$, $\mu = 10$. By (2.9) the nonlinear problem (7.5) is unstable. Observe that Λ^* defined by (4.5) is given by $\Lambda^* = 4$, and $\tilde{f}$ defined by (4.7) by

$$\tilde{f} = \begin{cases} 1 & \text{if } a \le 0, \\ 1/2 & \text{if } a > 0. \end{cases}$$

Therefore, with ν defined by (4.11), we have $a > \nu_\lambda(\mu)$, so by Theorem 5.1, we can choose an interval of values of $|\sigma|$ such that all solutions of (7.6) converge to zero exponentially fast, almost surely. Recall the subdivision of the (μ, a)-plane given in (1)-(3) in Proposition 4.2. We find ranges of the parameter $\alpha > 0$ such that

$a > f_2(\mu)$, or $f_3(\mu) < a < f_2(\mu)$ (where f_2, f_3 are given by (4.16), (4.17)), and hence use (4.21), (4.22) to determine the maximal interval of $|\sigma|$ for which solutions of (7.6) are stable.

By rearranging the inequality $a > f_2(\mu)$, we obtain a quadratic in α. Hence for

$$1 - \frac{\sqrt{2}}{2} < \alpha < 1 + \frac{\sqrt{2}}{2} \tag{7.7}$$

the point $(10, -1)$ (in the (μ, a) plane) satisfies $-1 > f_2(10)$. Rearranging the inequality $a > f_3(\mu)$ yields

$$\alpha > \frac{9 - \sqrt{65}}{4} \quad \text{or} \quad \alpha < \frac{9 + \sqrt{65}}{4},$$

and solving for $a < f_2(\mu)$ gives $\alpha < 1 - \sqrt{2}/2$ or $\alpha > 1 + \sqrt{2}/2$. Thus $f_3(\mu) < a < f_2(\mu)$ if and only if

$$\frac{9 - \sqrt{65}}{4} < \alpha < 1 - \frac{\sqrt{2}}{2} \quad \text{or} \quad 1 + \frac{\sqrt{2}}{2} < \alpha < \frac{9 + \sqrt{65}}{4}. \tag{7.8}$$

Appealing to (4.21), we see that the appropriate choice of $\sigma^2 > 0$ to stabilise (7.6) is given by

$$2(1 + 2\alpha) < \sigma^2 < \frac{2}{3}\left(11 - 2(\alpha - \alpha^{-1})\right)$$

for any α satisfying (7.7). This can be achieved for $\sigma^2 \in (3.17157, 8.82843)$.

Similarly, by (4.22), if we can choose α to satisfy (7.8) so that $\sigma^2 > 0$ is given by

$$2\sqrt{10 - 2\alpha^{-1}} - \sqrt{18 - 4(\alpha + \alpha^{-1})} < |\sigma| < 2\sqrt{10 - 2\alpha^{-1}} + \sqrt{18 - 4(\alpha + \alpha^{-1})}$$

then (7.6) can be stabilised. By (7.8), it follows that this can be achieved for $|\sigma| \in (1.7684, 8.9138)$.

Combining the analysis on the regions $a > f_2(\mu)$, $f_3(\mu) < a < f_2(\mu)$, we see that for any value of $\sigma^2 \in (3.127, 79.456)$ that all solutions of (7.6) satisfy

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t; \xi)| < 0, \quad \text{a.s.,}$$

while no solution of (7.5) converges to zero.

8. Finite Dimensional Problem

In this section, we consider how the analysis used to tackle the scalar problem can be adapted to deal with the stabilisation of the finite dimensional Itô-Volterra equation.

Suppose $f \in C(\mathbb{R}^d; \mathbb{R}^d)$ is Lipschitz continuous with a global linear bound, and with $f(0) = 0$. Suppose further that there exists $a > 0$ such that

$$\langle f(x), x \rangle \le -a\|x\|^2, \quad x \in \mathbb{R}^d. \tag{8.1}$$

Let $g \in C(\mathbb{R}^d; \mathbb{R}^r)$ be locally Lipschitz continuous and obeying a global linear bound

$$\|g(x)\| \le \overline{g}\|x\|, \quad x \in \mathbb{R}^d. \tag{8.2}$$

Finally, suppose there exists $\lambda > 0$, $\mu > 0$ such that $K \in C(\mathbb{R}^+; \mathbb{R}^d \times \mathbb{R}^r)$ satisfies

$$\|K(t)\| \le \lambda e^{-\mu t}, \quad t \ge 0. \tag{8.3}$$

In the above $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$, $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^d$ (or $\mathbb{R}^r$, as appropriate), and the norm in (8.3) is the standard operator norm on $\mathbb{R}^d \times \mathbb{R}^r$. Also, denote I_d as the $d \times d$ identity matrix.

We study stability of the following Itô-Volterra equation:

$$dX(t) = \left(f(X(t)) + \int_0^t K(t-s) g(X(s))\, ds \right) dt + \Sigma X(t)\, dB(t) \tag{8.4}$$

where $\Sigma = \sigma I_d$ and $\sigma \in \mathbb{R}$, and $X(0) = \xi \in \mathbb{R}^d$. Here, as before, B is standard one-dimensional Brownian motion, ξ is independent of B, with $\mathbb{E}[\|\xi\|^2] < \infty$. Under these conditions, there is a unique strong solution of (8.4). Moreover, if $X(0) = 0$, then $X(t) = 0$ for all $t \geq 0$ a.s.

Remark 8.1. We note for further reference that the statement of the stability theorem we obtain for equation (8.4) is identical for that we achieve for the equation

$$dX(t) = \left(f(t, X(t))) + \int_0^t K(s,t) g(s, X(s))\, ds \right) dt + \Sigma X(t)\, dB(t), \tag{8.5}$$

under the following restrictions on f, g, K: $\|K(s,t)\| \leq \lambda e^{-\mu(t-s)}$, $\|g(t,x)\| \leq \overline{g}\|x\|$, and $\langle x, f(t,x)\rangle \leq -\alpha\|x\|^2$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and some $\lambda, \mu, \overline{g}, \alpha > 0$.

To prove the stability result for (8.4), first define the process $Y(t) = \|X(t)\|^2$ and let $\tilde{\varphi}$ be given by $\tilde{\varphi}(0) = 1$ and

$$d\tilde{\varphi}(t) = 2\sigma\tilde{\varphi}(t)\, dB(t). \tag{8.6}$$

Also set $\tilde{Y}(t) = \tilde{\varphi}(t)^{-1} Y(t)$, $t \geq 0$. As Y has a semimartingle decomposition given by

$$dY(t) = \left(2\langle X(t), f(X(t))\rangle + 2\langle X(t), \int_0^t K(t-s) g(X(s))\, ds\rangle + \sigma^2 Y(t) \right) dt \\ + 2\sigma Y(t)\, dB(t), \tag{8.7}$$

by using (8.6) in conjunction with (8.7), we see that $(\tilde{Y}(t))_{t\geq 0}$ is a continuously differentiable process obeying

$$\tilde{Y}'(t) = \sigma^2 \tilde{Y}(t) - 2\langle X(t), f(X(t))\rangle \tilde{\varphi}(t)^{-1} + \tilde{\varphi}(t)^{-1} 2\langle X(t), \int_0^t K(t-s) g(X(s))\, ds\rangle. \tag{8.8}$$

Employing (8.1) gives

$$2\langle X(t), f(X(t))\rangle \tilde{\varphi}(t)^{-1} \leq -2a\tilde{Y}(t). \tag{8.9}$$

Using the Cauchy-Schwarz inequality (8.2), (8.3) and the inequality

$$2xy \leq \beta x^2 + \frac{1}{\beta} y^2 \quad (\beta > 0,\ x, y \geq 0)$$

we get

$$\begin{aligned} &\left| 2\langle X(t), \int_0^t K(t-s) g(X(s))\, ds\rangle \right| \\ &\leq 2\int_0^t \|X(t)\| \|K(t-s)\| \overline{g} \|X(s)\|\, ds \\ &\leq \lambda\overline{g} \int_0^t e^{-\mu(t-s)} \left(\beta\|X(t)\|^2 + \frac{1}{\beta}\|X(s)\|^2 \right) ds \\ &\leq \frac{\lambda\overline{g}}{\mu}\beta Y(t) + \frac{\lambda\overline{g}}{\beta}\int_0^t e^{-\mu(t-s)} Y(s)\, ds. \end{aligned} \tag{8.10}$$

Inserting (8.9) and (8.10) into (8.8) yields

$$\tilde{Y}'(t) \leq \left(\sigma^2 - 2a + \frac{\lambda\overline{g}}{\mu}\beta \right) \tilde{Y}(t) + \int_0^t \frac{\lambda\overline{g}}{\beta} e^{-\mu(t-s)} \tilde{\varphi}(t)^{-1} \tilde{\varphi}(s) \tilde{Y}(s)\, ds. \tag{8.11}$$

Next define $(\tilde{Z}(t))_{t\geq 0}$ according to $\tilde{Z}(0) = \tilde{Y}(0)$ and

$$\tilde{Z}'(t) = \left(\sigma^2 - 2a + \frac{\lambda\overline{g}}{\mu}\beta \right) \tilde{Z}(t) + \int_0^t \frac{\lambda\overline{g}}{\beta} e^{-\mu(t-s)} \tilde{\varphi}(t)^{-1} \tilde{\varphi}(s) \tilde{Y}(s)\, ds.$$

By applying the comparison principle pathwise, we see that $\tilde{Y}(t) \leq \tilde{Z}(t)$ for $t \geq 0$, a.s. Now let $Z(t) = \tilde{\varphi}(t)\tilde{Z}(t)$; then Z obeys
(8.12)

$$dZ(t) = \left[\left(\sigma^2 - 2a + \frac{\lambda \overline{g}}{\mu}\beta\right) Z(t) + \int_0^t \frac{\lambda \overline{g}}{\beta} e^{-\mu(t-s)} Z(s)\, ds\right] dt + 2\sigma Z(t)\, dB(t),$$

and so we obtain

$$\|X(t)\|^2 = Y(t) = \tilde{\varphi}(t)\tilde{Y}(t) \leq \tilde{\varphi}(t)\tilde{Z}(t) = Z(t).$$

Therefore, if we can show that there exists $\delta > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log Z(t) \leq -\delta_0, \quad \text{a.s.} \tag{8.13}$$

it is automatically true that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| \leq -\frac{\delta_0}{2}, \quad \text{a.s.}$$

Note further that $Z(0) = \|\xi\|^2$ and for any $p > 0$ that $\mathbb{E}[\|X(t)\|^p] \leq \mathbb{E}[Z(t)^{p/2}]$, so we may equally obtain stability results in p-th mean (for sufficiently small p) as before.

The above argument supplies the proof of the following Theorem.

Theorem 8.2. *Suppose f, g are locally Lipschitz continuous and globally linearly bounded with $f(0) = 0$, $g(0) = 0$, and satisfy (8.1), (8.2). If K is continuous and satisfies (8.3), then the solution of (8.4) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t;\xi)\| \leq -\frac{\delta_0}{2}, \quad a.s.,$$

for some $\delta_0 > 0$, provided that the solution of (8.12) satisfies (8.13). Moreover, under these conditions, there exists $p^* > 0$ such that for all $p < p^*$ there is $\delta_p > 0$ such that

$$\mathbb{E}[\|X(t;\xi)\|^p] \leq C\mathbb{E}[\|\xi\|^p]e^{-\delta_p t}, \quad t \geq 0$$

where C is a positive constant independent of p and ξ.

Showing that the solution of the Itô-Volterra equation (8.12) satisfies (8.13) for some $\delta_0 > 0$ is the subject of an earlier section of the paper, and, as before, our interest lies in the stabilising effect of a noise perturbation. In (8.12), however, unlike the earlier problem studied, the intensity of the noise perturbation arises in both drift and diffusion terms of the auxiliary Itô-Volterra equation (8.12), so although the method of proof is identical in spirit to previous results, the calculation must be done afresh.

Lemma 8.3. *Suppose $\overline{g}, \mu, \lambda > 0$, and define*

$$\nu(\lambda, \overline{g}, \mu) = \begin{cases} \overline{g}\lambda/\mu & \text{for } \mu \in (0, \sqrt{3\overline{g}\lambda} \\ \frac{3}{2}\left(\frac{(\overline{g}\lambda)^2}{3\mu}\right)^{1/3} - \frac{1}{6}\mu & \text{for } \mu \geq \sqrt{3\overline{g}\lambda}. \end{cases} \tag{8.14}$$

If $a > \nu(\lambda, \overline{g}, \mu)$, there exists a non-empty interval $I_{a,\mu,\overline{g},\lambda} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\overline{g},\lambda}$ there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \overline{g}, \lambda) > 0$ such that $\delta_0^* < \delta_0$, where δ_0 is given by (8.13).

Proof. To obtain sufficient conditions under which $\delta_0 > 0$ note that

$$-\delta_0 \leq \max_{x\in[0,1]} H(x),$$

where

$$H(x) = -4\sigma^2 x^2 + x\left(3\sigma^2 - 2a + \frac{\overline{g}\beta\lambda}{\mu} + \frac{\overline{g}\lambda}{\beta}(\alpha - \alpha^{-1}) + \mu\right) + \frac{1}{\alpha}\sqrt{\frac{\overline{g}\lambda}{\beta}} - \mu,$$

and α, β are positive constants that are chosen so as to obtain the best sufficient condition on the parameters under which stability is guaranteed. As before, we note that $\delta_0^* > 0$ if one of the following hold:

(i) $H(0) < 0$, $H'(0) < 0$;
(ii) $H(1) < 0$, $H'(1) > 0$;
(iii) There exists $x^* \in [0, 1]$ such that $H'(x^*) = 0$ and $H(x^*) < 0$.

Now, define $\mu^* = \alpha^{-1}\sqrt{\overline{g}\lambda/\beta}$, and the functions

$$f_1(\mu, \alpha, \beta) = \frac{1}{2}\left(\overline{g}\beta\frac{\lambda}{\mu} + \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha - \alpha^{-1} + \mu\right),$$

$$f_2(\mu, \alpha, \beta) = \frac{1}{2}\left(\overline{g}\beta\frac{\lambda}{\mu} + \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha + \frac{1}{4}\alpha^{-1} - \mu/4\right),$$

$$f_3(\mu, \alpha, \beta) = \frac{1}{2}\left(\overline{g}\beta\frac{\lambda}{\mu} + \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha + \frac{1}{3}\alpha^{-1} - \mu/3\right),$$

for $\mu > \mu^*$. Note that $f_1(\mu, \alpha, \beta) > f_2(\mu, \alpha, \beta) > f_3(\mu, \alpha, \beta)$ for fixed $\alpha, \beta > 0$ and $\mu > \mu^*$, while

$$f_j(\mu^*) = \frac{1}{2}\alpha\sqrt{\overline{g}\lambda}(\beta^{-1/2} + \beta^{3/2}), \quad j = 1, 2, 3.$$

Now, consider the parameter regions

(1) $a > f_1(\mu, \alpha, \beta)$, $\mu > \mu^*$;
(1') $a = f_1(\mu, \alpha, \beta)$, $\mu > \mu^*$;
(2) $f_2(\mu, \alpha, \beta) < a < f_1(\mu, \alpha, \beta)$, $\mu > \mu^*$;
(3) $f_3(\mu, \alpha, \beta) < a \le f_2(\mu, \alpha, \beta)$, $\mu > \mu^*$.

It transpires that for the appropriate ranges of σ that (1) implies (i), (2) implies (ii), (3) implies (iii), while (1') gives rise to a special case of (iii). We omit the calculations which justify this statement. In order that the results may be used in practice, we give here, without justification, the appropriate ranges of σ:

$$0 < \sigma^2 < \frac{1}{3}(2a - \overline{g}\beta\frac{\lambda}{\mu} - \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha - \alpha^{-1}) - \mu) \quad \text{for (1)}, \tag{8.15}$$

$$0 < \sigma^2 < \frac{16}{9}(\mu - \frac{1}{\alpha}\sqrt{\frac{\overline{g}\lambda}{\beta}}) \quad \text{for (1')}, \tag{8.16}$$

$$-2a + \overline{g}\beta\frac{\lambda}{\mu} < \sigma^2 < \frac{1}{5}(-2a + \overline{g}\beta\frac{\lambda}{\mu} + \mu + \sqrt{\frac{\overline{g}\lambda}{\beta}}(\alpha - \alpha^{-1}) \quad \text{for (2)}, \tag{8.17}$$

$$\sigma_-^2 < \sigma^2 < \sigma_+^2 \quad \text{for (3)} \tag{8.18}$$

where $0 < \sigma_- < \sigma_+$ are given by

$$\sigma_\pm = \frac{2}{3}\sqrt{\mu - \mu^*} \pm \frac{1}{3}\sqrt{6(a - f_3(\mu, \alpha, \beta))}.$$

Now fix $\alpha, \beta > 0$ and $\overline{g}, \lambda > 0$, and define

$$S_{\alpha,\beta} = \{(\mu, a) \in \mathbb{R}^+ \times \mathbb{R} : \mu > \mu^*(\alpha, \beta), a > f_3(\mu, \alpha, \beta)\}.$$

Then the following is true: if $(\mu, a) \in S_{\alpha,\beta}$ for some $\alpha, \beta > 0$ there exists a non-empty interval $I_{a,\mu,\overline{g},\lambda,\alpha,\beta} \subset \mathbb{R}^+$ such that for all $|\sigma| \in I_{a,\mu,\overline{g},\lambda,\alpha,\beta}$ we have $H(x) < 0$ for all $x \in [0, 1]$. (The intervals $I_{a,\mu,\overline{g},\lambda,\alpha,\beta}$ in cases (1), (1'), (2), (3) are determined in (8.15), (8.16), (8.17), (8.18), respectively.) Now, suppose $0 < \mu \le \sqrt{3\overline{g}\lambda}$ and $a > \overline{g}\lambda/\mu$. Let $\beta = 1$ and note there exists $\alpha > 0$ such that

$$\frac{1}{\mu}\sqrt{\overline{g}\lambda} < \alpha < \frac{a}{\sqrt{\overline{g}\lambda}}.$$

Hence $(\mu, a) \in S_{\alpha,\beta}$. Next, suppose that $\mu > \sqrt{3\overline{g}\lambda}$ and

$$a > \frac{3}{2}\left(\frac{(\overline{g}\lambda)^2}{3\mu}\right)^{1/3} - \frac{1}{6}\mu.$$

Now let $\alpha = 1/\sqrt{3}$, and $\beta^{3/2} = \mu/(3\overline{g}\lambda)^{1/2}$. Then $a > f_3(\mu, \alpha, \beta)$ and so $(\mu, a) \in S_{\alpha,\beta}$. Fixing the $\alpha-$, $\beta-$dependence in terms of $a, \mu, \overline{g}, \lambda$ above yields the required intervals $I_{a,\mu,\overline{g},\lambda} = I_{a,\mu,\overline{g},\lambda,\alpha(a,\mu,\overline{g},\lambda),\beta(a,\mu,\overline{g},\lambda)}$, with $-\delta_0^* = \max_{x\in[0,1]} H(x) < 0$. □

This gives the desired stabilisation result.

Theorem 8.4. *Suppose that f, g are globally linearly bounded and locally Lipschitz continuous, satisfy $f(0) = 0$, $g(0) = 0$, and obey (8.1), (8.2). Let $K \in C(\mathbb{R}^+; M_{d,r}(\mathbb{R}))$ satisfy (8.3), and $\Sigma = \sigma I_d$. Suppose that $\nu(\mu, \lambda, \overline{g})$ be defined by (8.14) and $a > \nu(\mu, \lambda, \overline{g})$. Then:*

(i) *There exists a non-empty interval $I_{a,\mu,\lambda,\overline{g}} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{a,\mu,\lambda,\overline{g}}$ there exists $\delta_0^* = \delta_0^*(|\sigma|, a, \mu, \lambda, \overline{g}) > 0$ so that all solutions of (8.4) satisfy*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t;\xi)\| \le -\delta_0^*, \quad a.s.$$

(ii) *Moreover, there exists $p^* > 0$ such that for all $p < p^*$, there is $\delta_p > 0$ such that*

$$\mathbb{E}[\|X(t;\xi)\|^p] \le C\mathbb{E}[\|\xi\|^p]e^{-\delta_p t}, \quad t \ge 0,$$

where C is a positive constant independent of p and ξ.

8.1. **Stabilisation without a negative definite assumption on f.** It is still possible to prove a stabilisation result for the finite dimensional Volterra equation

$$x'(t) = f(x(t)) + \int_0^t K(t-s)g(x(s))\, ds \tag{8.19}$$

even when f does not satisfy the negative definite assumption (8.1) posited earlier in this section. In the sequel, we relax the hypothesis (8.1) on f in favour of requiring that f obeys a global linear bound of the form:

$$\|f(x)\| \le \overline{f}\|x\|, \quad x \in \mathbb{R}^d \tag{8.20}$$

Again, we study the asymptotic behaviour of (8.4) under the assumptions (8.20), (8.2), (8.3), assuming as before, that f, g are locally Lipschitz continuous, and K continuous. To this end, we introduce the $M_{d,d}(\mathbb{R}^+)$-valued process $(\Phi(t))_{t\ge 0}$ determined by

$$d\Phi(t) = \sigma\Phi(t)\, dB(t), \quad t \ge 0$$

and $\Phi(0) = I_d$. Hence $\Phi(t) = \varphi(t)I_d$ where $\varphi(0) = 1$ and

$$d\varphi(t) = \sigma\varphi(t)\, dB(t)$$

is a scalar geometric Brownian motion. Next, define $Y(t) = \Phi(t)^{-1}X(t)$ for $t \ge 0$; this process is continuously differentiable on $\mathbb{R}^+$ and satisfies

$$Y'(t) = \Phi(t)^{-1}\left(f(\Phi(t)Y(t)) + \int_0^t K(t-s)g(\Phi(s)Y(s))\, ds\right).$$

Employing (8.20), (8.2), (8.3) and the fact that $\|\Phi(t)\| = \varphi(t)$, $\|\Phi(t)^{-1}\| = \varphi(t)^{-1}$, we get

$$\|Y'(t)\| \leq \overline{f}\|Y(t)\| + \int_0^t \lambda e^{-\mu(t-s)}\overline{g}\varphi(t)^{-1}\varphi(s)\|Y(s)\|\,ds,$$

so for $t \geq 0$

$$D_+\|Y(t)\| \leq \overline{f}\|Y(t)\| + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}\varphi(t)^{-1}\varphi(s)\|Y(s)\|\,ds.$$

Therefore, if $\tilde{Z}(0) = \|\xi\|$ and $(\tilde{Z}(t))_{t\geq 0}$ satisfies

$$\tilde{Z}'(t) = \overline{f}\tilde{Z}(t) + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}\varphi(t)^{-1}\varphi(s)\tilde{Z}(s)\,ds,$$

the comparison principle implies that $\|Y(t)\| \leq \tilde{Z}(t)$ for all $t \geq 0$, a.s. Next observe that $(Z(t))_{t\geq 0}$ defined by $Z(t) = \varphi(t)\tilde{Z}(t)$ satisfies

$$dZ(t) = \left(\overline{f}Z(t) + \int_0^t \lambda\overline{g}e^{-\mu(t-s)}Z(s)\,ds\right)dt + \sigma Z(t)\,dB(t). \tag{8.21}$$

Since

$$\|X(t)\| = \|\Phi(t)Y(t)\| \leq \varphi(t)\|Y(t)\| \leq \varphi(t)\tilde{Z}(t) = Z(t),$$

it follows that there exists $\delta_0 > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log\|X(t;\xi)\| \leq -\delta_0 \quad \text{a.s.}$$

providing that Z satisfying (8.21) obeys

$$\limsup_{t\to\infty} \frac{1}{t}\log Z(t) \leq -\delta_0, \quad \text{a.s.} \tag{8.22}$$

Applying Theorem 5.2 to the equation (8.21), it is seen that (8.22) holds for some range of σ once

$$\overline{f} \leq \mu - 2\sqrt{\lambda\overline{g}}, \quad \mu \geq 2\sqrt{\lambda\overline{g}}.$$

Therefore, we have obtained the following result.

Theorem 8.5. *Suppose that f, g satisfy (8.20), (8.2) and are locally Lipschitz continuous. Let $K \in C(\mathbb{R}^+; M_{d,r}(\mathbb{R}))$ satisfy (8.3), and suppose that $\Sigma = \sigma I_d$. If $0 \leq \overline{f} \leq \mu - 2\sqrt{\lambda\overline{g}}$, then:*

(i) *There exists a non-empty interval $I_{f,g,K} \subset \mathbb{R}^+$ such that for $|\sigma| \in I_{f,g,K}$ there is $\delta_0^* = \delta_0^*(|\sigma|, f, g, K) > 0$ so that all solutions of (8.4) satisfy*

$$\limsup_{t\to\infty} \frac{1}{t}\log\|X(t;\xi)\| \leq -\delta_0^*, \quad \text{a.s.}$$

(ii) *Moreover, under these conditions, there exists $p^* > 0$ such that for all $p < p^*$ there is $\delta_p > 0$ such that*

$$\mathbb{E}[\|X(t;\xi)\|^p] \leq C\mathbb{E}[\|\xi\|^p]e^{-\delta_p t}, \quad t \geq 0$$

where C is a positive constant independent of p and ξ.

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