MS308: Stochastic Modelling

VI. Brownian Motion, Diffusions and Stochastic Calculus

1. (Martingale Property of B_t^3)

(a) If $(B_t)_{t\geq 0}$ is a standard Brownian motion with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, calculate

$$\mathbb{E}[B_t^3|\mathcal{F}_s] \quad \text{for } t > s,$$

and hence show that $M_t = B_t^3 - 3tB_t$ is an $\mathcal{F}_t$-martingale.

(b) Use Itô's rule and the stochastic version of the product rule on B_t^3 and tB_t respectively to show that M_t is a martingale.

2. (Kolmogorov Equations and Stationary density of O-U process)

Suppose that $\{X_t : t \geq 0\}$ is a time-homogeneous diffusion with drift $\mu(x) = -\gamma x$ and diffusion coefficient $\sigma^2(x) = \sigma^2$, where $\gamma > 0$.

(a) Write down the Forward Kolmogorov p.d.e. (Fokker-Planck equation) and the ordinary differential equation satisfied by the stationary probability density $\pi(y)$.

(b) Verify by direct computation that

$$\pi(y) = g_{\sigma^2/2\gamma}(y)$$

where g_v is the density of a Gaussian distribution with zero mean and variance v.

3. (O-U process via Stochastic Differential Equations)

Let $\{X_t : t \geq 0\}$ be a diffusion process characterised by the stochastic differential equation

$$dX_t = -\gamma X_t\, dt + \sigma\, dB_t,$$

where γ, σ are as in Q2 above. $\{X_t : t \geq 0\}$ is the same Ornstein-Uhlenbeck process as in Q2 above. We show here that it has the same long run distribution as the diffusion process in Q2.

(a) If $U_t = e^{\gamma t} X_t$, show that $dU_t = \sigma e^{\gamma t}\, dB_t$.

(b) What is the distribution of X_t, for fixed $t > 0$, if X_0 is a deterministic constant? Show that

$$\lim_{t\to\infty} \mathbb{P}[X_t \leq x] = \int_{-\infty}^{x} \pi(y)\, dy$$

where π is defined in Q2 above, and $x \in \mathbb{R}$.

4. (Stock price model)

Suppose the price of a share of a company is given at time t years by S_t, where S_t is determined by the stochastic differential equation:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t.$$

(a) Find the S.D.E. for $X_t = \log(S_t/S_0)$ using Itô's rule.

(b) Establish from this that

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}.$$

(c) If the expected instantaneous growth p.a. of the stock is 12% ($\mu = 0.12$), and the instantaneous volatility p.a. is 9%, ($\sigma = 0.09$) what is the probability that the price more than doubles in 7 years?